

# Analysis of Finite Element Methods for Vector Laplacians on Surfaces <sup>\*</sup>

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## Abstract

We develop a finite element method for the vector Laplacian based on the covariant derivative of tangential vector fields on surfaces embedded in  $\mathbb{R}^3$ . Closely related operators arise in models of flow on surfaces as well as elastic membranes and shells. The method is based on standard continuous parametric Lagrange elements with one order higher polynomial degree for the mapping. The tangent condition is weakly enforced using a penalization term. We derive error estimates that takes the approximation of both the geometry of the surface and the solution to the partial differential equation into account. We also present numerical results that verify our theoretical findings.

**Subject Classification Codes:** 65M60, 65N15, 65N30.

**Keywords:** surface finite element methods, vector Laplacian, a priori error estimates.

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# Contents

|          |  |           |
|----------|--|-----------|
| <b>1</b> | <b>Introduction</b>  | <b>3</b>  |
| <b>2</b> | <b>Vector Laplacians on a Surface</b>                      | <b>3</b>  |
| 2.1      | The Surface . . . . .                                      | 3         |
| 2.2      | Tangential Calculus . . . . .                              | 4         |
| 2.3      | Function Spaces . . . . .                                  | 6         |
| 2.4      | Vector Laplacians . . . . .                                | 6         |
| <b>3</b> | <b>The Finite Element Method</b>                           | <b>7</b>  |
| 3.1      | Triangulation of the Surface . . . . .                     | 7         |
| 3.2      | Parametric Finite Element Spaces . . . . .                 | 7         |
| 3.3      | Interpolation . . . . .                                    | 7         |
| 3.4      | Formulation of the Method . . . . .                        | 9         |
| <b>4</b> | <b>Preliminary Results</b>                                 | <b>9</b>  |
| 4.1      | Extension and Lifting of Functions . . . . .               | 9         |
| 4.2      | Extension and Lifting of Vector Valued Functions . . . . . | 10        |
| 4.3      | Estimates Related to $B$ . . . . .                         | 10        |
| 4.4      | Norm Equivalences . . . . .                                | 11        |
| <b>5</b> | <b>Error Estimates</b>                                     | <b>12</b> |
| 5.1      | Norms . . . . .  | 12        |
| 5.2      | Coercivity and Continuity . . . . .                        | 12        |
| 5.3      | Basic Lemmas . . . . .                                     | 12        |
| 5.4      | Interpolation . . . . .                                    | 19        |
| 5.5      | Estimates of Geometric Errors . . . . .                    | 19        |
| 5.6      | Error Estimates . . . . .                                  | 21        |
| <b>6</b> | <b>Numerical Results</b>                                   | <b>25</b> |
| 6.1      | Model Problem and Numerical Example . . . . .              | 25        |
| 6.2      | Convergence . . . . .                                      | 26        |

# 1 Introduction

In this contribution we develop a finite element method for the vector Laplacian on a surface. The vector Laplacian we consider is a second order elliptic operator based on covariant derivatives acting on tangential vector fields, in contrast to the Hodge Laplace operator which is based on exterior calculus, see [12]. The method is based on a triangulation of the surface with polynomials of order  $k_g$  and the finite element space is the standard vector valued continuous parametric Lagrange elements of order  $k_u$ . The tangential condition is enforced weakly using a suitable penalty term, similar to our work on the Darcy problem, see [10]. Note, however, that the Darcy problem does not involve any gradients of the velocity vector and is therefore easier to deal with. This approach leads to a convenient implementation without the need for special finite element spaces.

We prove a priori error estimates in the energy and  $L^2$  norm and we find that in order to obtain optimal order convergence in  $L^2$  it is necessary to take the geometry approximation order one step higher than the order of the finite element space, i.e.  $k_g = k_u + 1$ . This is due to the fact that the covariant derivative is obtained by a projecting the componentwise directional derivative onto the tangent plane and the approximation order of the projection is only  $h^{k_g}$ .

Finite elements for partial differential equations on surfaces is now a rapidly developing field that originates from the seminal work of Dziuk [5] where surface finite elements for the Laplace–Beltrami operator was first developed. Most of the research is, however, focused on problems with scalar unknowns, see the recent review article [6] and the references therein, which simplifies the differential calculus since the covariant derivative of a vector field or more generally a tensor field is not needed. Models of flow on surfaces as well as membranes and shells, however, involve vector unknowns, see for instance [9] (linear) and [11] (nonlinear), for membrane models formulated using the same approach as used in this paper. Furthermore, we employ higher order elements similar to the approach presented in [3, 10, 13, 14].

The outline of the paper is as follows: In Section 2 we introduce the vector Laplacian, in Section 3 we introduce the finite element method, in Section 4 we recall some basic results regarding lifting and extension of functions between the discrete and continuous surfaces, in Section 5 we derive a sequence of necessary lemmas leading up to the a priori error estimate, and finally in Section 6 we present numerical examples confirming our theoretical findings.

## 2 Vector Laplacians on a Surface

### 2.1 The Surface

Let  $\Gamma$  be a compact surface embedded in  $\mathbb{R}^3$  without boundary and let  $\rho$  be the signed distance function, negative on the inside and positive on the outside. Let  $p : \mathbb{R}^3 \rightarrow \Gamma$  be the closest point mapping onto  $\Gamma$ . Then there is a  $\delta_0 > 0$  such that  $p$  maps each point in  $U_{\delta_0}(\Gamma)$  to precisely one point on  $\Gamma$ , where  $U_\delta(\Gamma) = \{x \in \mathbb{R}^3 : |\rho(x)| < \delta\}$  is the open

tubular neighborhood of  $\Gamma$  of thickness  $\delta > 0$ . The exterior unit normal to the surface is given by  $n = \nabla \rho$ .

## 2.2 Tangential Calculus

For each function  $u : \Gamma \rightarrow \mathbb{R}^m$ ,  $m = 1, 2, \dots$ , we define the componentwise extension  $u^e$  to the neighborhood  $U_{\delta_0}(\Gamma)$  by the pull back  $u^e = u \circ p$ . Let  $\{e_i \in \mathbb{R}^3\}_{i=1}^3$  be the fixed Euclidean basis to the embedding space  $\mathbb{R}^3$  and let  $P = I - n \otimes n$  be the projection onto the tangential plane  $T_x(\Gamma)$ . The projected Euclidean basis  $\{p_i = P e_i : \Gamma \rightarrow T_x(\Gamma)\}_{i=1}^3$  spans the tangential plane  $T_x(\Gamma)$  but does not constitute a basis to  $T_x(\Gamma)$  as the vectors in the set are linearly dependent. We will also make use of  $Q = n \otimes n$ , i.e. the projection onto the normal line. In the paragraphs below we let  $a$  be a tangential vector field on  $\Gamma$ , i.e.

$$a : \Gamma \ni x \rightarrow \sum_{i=1}^3 (a_i p_i)(x) \in T_x(\Gamma) \quad (2.1)$$

where  $a_i : \Gamma \rightarrow \mathbb{R}$  are coefficients.

**Tangential Derivatives.** Let  $u$  be a vector field on  $\Gamma$

$$u : \Gamma \ni x \rightarrow \sum_{i=1}^3 u_i(x) e_i \in \mathbb{R}^3 \quad (2.2)$$

We define the directional derivative of  $u$  in the direction of a tangential vector field  $a$  as

$$\partial_a u = (a \cdot \nabla) u^e = (u^e \otimes \nabla) \cdot a \quad (2.3)$$

By introducing the tangential del operator  $\nabla_\Gamma = [\partial_{p_1}, \partial_{p_2}, \partial_{p_3}]^T$  we can collect all first tangential derivatives in the definition for the gradient of a vector field

$$u \otimes \nabla_\Gamma = [\partial_{p_1} u, \partial_{p_2} u, \partial_{p_3} u] = (u^e \otimes \nabla) P \quad (2.4)$$

and we note that  $\partial_a u = (u \otimes \nabla_\Gamma) \cdot a$ .

More generally, derivatives of order  $m$  are represented as the following  $(m + 1, 0)$  tensor expanded in the Euclidean basis

$$u \otimes \nabla_\Gamma^m = u \otimes \underbrace{\nabla_\Gamma \otimes \dots \otimes \nabla_\Gamma}_{m \text{ gradients}} = \sum_{i_0, \dots, i_m=1}^3 u_{i_0 \dots i_m} (e_{i_0} \otimes \dots \otimes e_{i_m}) \quad (2.5)$$

where the coefficients are given by  $u_{i_0 \dots i_m} = \partial_{p_m} \partial_{p_{m-1}} \dots \partial_{p_1} u_{i_0}$ . As the Euclidean basis is orthonormal and we inherit the Euclidean inner product from the embedding space the dual and primal basis are the same, i.e.  $\{e^i = e_i\}_{i=1}^3$ . We can thus retrieve the coefficients from the expanded tensor (2.5) by the contraction

$$u_{i_0 \dots i_m} = (u \otimes \nabla_\Gamma^m) : (e^{i_0} \otimes \dots \otimes e^{i_m}) \quad (2.6)$$

where the contraction operator ":" is defined

$$(a_1 \otimes \dots \otimes a_m) : (b^1 \otimes \dots \otimes b^m) = (a_1 \cdot b^1) \dots (a_m \cdot b^m) \quad (2.7)$$

**Covariant Derivatives.** Let  $u$  be a tangential vector field on  $\Gamma$

$$u : \Gamma \ni x \rightarrow \sum_{i=1}^3 (u_i p_i)(x) \in T_x(\Gamma) \quad (2.8)$$

The covariant derivative of a tangential vector field  $u$  in the direction of a tangential vector field  $a$  can, in the embedded setting, be expressed

$$D_a u = P \partial_a u = P \partial_a \sum_{i=1}^3 u_i p_i = \sum_{i=1}^3 (\partial_a u_i) p_i + u_i P(\partial_a p_i) \quad (2.9)$$

where we from the product rule have that the covariant derivative includes a lower order term multiplied by a projected derivative of a tangent basis vector. Expanding  $P(\partial_a p_i)$  in some tangent basis yields a tensor whose coefficients correspond to the Christoffel symbols of the Levi-Civita connection. We collect all first covariant derivatives in the tensor of covariant derivatives

$$D_\Gamma u = [D_{p_1} u, D_{p_2} u, D_{p_3} u] = P(u \otimes \nabla_\Gamma) = P(u^e \otimes \nabla) P \quad (2.10)$$

and we note that  $D_a u = (D_\Gamma u) \cdot a = P(u \otimes \nabla_\Gamma) \cdot a$ . This tensor, in contrast to  $u \otimes \nabla_\Gamma$ , is a tangential tensor. The symmetric part of the tensor of covariant derivatives is defined by

$$\epsilon_\Gamma(u) = \frac{1}{2} (D_\Gamma u + (D_\Gamma u)^T) \quad (2.11)$$

which is the tangential strain tensor used in modeling of solids and fluids, see [9].

All tensors can be represented in a basis for the embedding space, in particular as an expansion in the Euclidean basis  $\{e_i\}_{i=1}^3$ . We construct the projection of a  $(m, 0)$  tensor as

$$\text{proj}_\Gamma(X) = \sum_{i_0, \dots, i_m=1}^3 (X : (e^{i_0} \otimes \dots \otimes e^{i_m})) (p_{i_0} \otimes \dots \otimes p_{i_m}) \quad (2.12)$$

where we first retrieve the coefficients of the expansion in the Euclidean basis and then expand the tensor in the projected Euclidean basis. The covariant derivative of a tangential tensor, i.e.  $X = \text{proj}_\Gamma(X) = \sum X_{i_0 \dots i_m} (p_{i_0} \otimes \dots \otimes p_{i_m})$ , is

$$D_a X = \sum_{i_0, \dots, i_m=1}^3 (\partial_a X_{i_0 \dots i_m}) (p_{i_0} \otimes \dots \otimes p_{i_m}) + X_{i_0 \dots i_m} \text{proj}_\Gamma(\partial_a (p_{i_0} \otimes \dots \otimes p_{i_m})) \quad (2.13)$$

We collect all first covariant derivatives of  $X$  in the tensor

$$D_\Gamma X = \sum_{j=1}^3 (D_{p_j} X) \otimes p_j \quad (2.14)$$

and note  $D_a X = (D_\Gamma X) \cdot a$ . By iterating this definition we can represent covariant derivatives of order  $m$  as

$$(D_\Gamma)^m u = \underbrace{D_\Gamma \dots D_\Gamma}_m u \quad (2.15)$$

$m$  covariant derivatives

## 2.3 Function Spaces

Let  $(\cdot, \cdot)_\omega$  and  $\|\cdot\|_{L^2(\omega)}$  denote the usual  $L^2$  inner product and norm on  $\omega$  and let  $\|\cdot\|_{L^\infty(\omega)}$  denote the usual  $L^\infty$  norm on  $\omega$ . We define the following Sobolev spaces on  $\Gamma$ :

- $H^s(\omega)$ , with  $\omega \subset \Gamma$ , denotes the standard Sobolev spaces of scalar or vector valued functions with componentwise derivatives and norm

$$\|v\|_{H^s(\omega)}^2 = \sum_{j=0}^s \|v \otimes \nabla_\Gamma^j\|_{L^2(\omega)}^2 \quad (2.16)$$

- $H_{\text{tan}}^s(\omega)$ , with  $\omega \subset \Gamma$ , denotes the Sobolev space of tangential vector fields with covariant derivatives and norm

$$\|v_t\|_{H_{\text{tan}}^s(\omega)}^2 = \sum_{j=0}^s \|(D_\Gamma)^j v_t\|_{L^2(\omega)}^2 \quad (2.17)$$

We employ the standard notation  $L^2(\omega) = H^0(\omega)$  and  $\|v\|_{L^2(\omega)} = \|v\|_\omega$ .

## 2.4 Vector Laplacians

We consider the variational problem: Find  $u \in H_{\text{tan}}^1(\Gamma)$  such that

$$a(u, v) = l(v) \quad \forall v \in H_{\text{tan}}^1(\Gamma) \quad (2.18)$$

where the forms are given by

$$a(u, v) = (D_\Gamma(u), D_\Gamma(v))_\Gamma \quad l(v) = (f, v)_\Gamma \quad (2.19)$$

where  $f$  is a given tangential vector field in  $H_{\text{tan}}^{-1}(\Gamma)$ . Using the Poincaré inequality (see Lemma 5.3 below) together with the Lax–Milgram lemma we conclude that this problem has a unique solution  $u \in H_{\text{tan}}^1(\Gamma)$ . Furthermore, for smooth surfaces we have the following elliptic shift property

$$\|u\|_{H_{\text{tan}}^{s+2}(\Gamma)} \leq C \|f\|_{H_{\text{tan}}^s(\Gamma)}, \quad -1 \leq s \quad (2.20)$$

with a constant  $C = C(\Gamma, s)$ .

We will also briefly consider the corresponding problem based on the symmetric part of the covariant derivative: Find  $u \in H_{\text{tan}}^1(\Gamma)$  such that

$$a_{\text{sym}}(u, v) = l(v) \quad \forall v \in H_{\text{tan}}^1(\Gamma) \quad (2.21)$$

where the bilinear form is given by

$$a_{\text{sym}}(u, v) = (\epsilon_\Gamma(u), \epsilon_\Gamma(v))_\Gamma \quad (2.22)$$

For simplicity we will however here focus our presentation on the standard formulation (2.18). In Remark 5.2 we comment on the extensions in the error analysis necessary to handle the symmetric formulation.

### 3 The Finite Element Method

#### 3.1 Triangulation of the Surface

**Parametric Triangulated Surfaces.** Let  $\hat{K} \subset \mathbb{R}^2$  be a reference triangle and let  $P_{k_g}(\hat{K})$  be the space of polynomials of order less or equal to  $k_g$  defined on  $\hat{K}$ . Let  $\Gamma_{h,k_g}$  be a triangulated surface in  $\mathbb{R}^3$  with quasi uniform triangulation  $\mathcal{K}_{h,k_g}$  and mesh parameter  $h \in (0, h_0]$  such that each triangle  $K$  can be described via a mapping  $F_{K,k_g} : \hat{K} \rightarrow K$  where  $F_{K,k_g} \in [P_{k_g}(\hat{K})]^3$ . Let  $n_{h,k_g}$  be the elementwise defined normal to  $\Gamma_{h,k_g}$ . For simplicity we use the notation  $\mathcal{K}_h = \mathcal{K}_{h,k_g}$ ,  $\Gamma_h = \Gamma_{h,k_g}$  and  $n_h = n_{h,k_g}$ .

We let geometric properties of  $\Gamma_h$  be indicated by subscript  $h$ , for example the discrete curvature tensor  $\kappa_h$  and the projections  $P_h = I - Q_h$  and  $Q_h = n_h \otimes n_h$  onto the discrete tangential plane respectively onto the discrete normal line.

**Geometry Approximation Assumption.** We assume that the family  $\{\Gamma_{h,k_g}, h \in (0, h_0]\}$  approximates  $\Gamma$  in the following ways:

- $\Gamma_{h,k_g} \subset U_{\delta_0}(\Gamma)$  and  $p : \Gamma_{h,k_g} \rightarrow \Gamma$  is a bijection.
- The following estimates hold:

$$\|p\|_{L^\infty(\Gamma_{h,k_g})} \lesssim h^{k_g+1}, \quad \|n \circ p - n_h\|_{L^\infty(\Gamma_{h,k_g})} \lesssim h^{k_g} \quad (3.1)$$

Here and below we let  $a \lesssim b$  denote  $a \leq Cb$  with a constant  $C$  independent of the mesh parameter  $h$ .

From (3.1) we can derive bounds for approximation of other geometric quantities, for example  $\|P - P_h\|_{L^\infty(\Gamma_h)} \lesssim h^{k_g}$  and  $\|P \cdot n_h\|_{L^\infty(\Gamma_h)} \lesssim h^{k_g}$ , see eg. [3].

#### 3.2 Parametric Finite Element Spaces

Let

$$V_{h,k_u,k_g} = \{v : v|_K \circ F_{K,k_g} \in P_k(\hat{K}), \forall K \in \mathcal{K}_{h,k_g}; v \in C^0(\Gamma_h)\} \quad (3.2)$$

be the space of parametric continuous piecewise polynomials of order  $k_u$  mapped with a mapping of order  $k_g$ . For brevity we use the simplified notation

$$V_h = [V_{h,k_u,k_g}]^3 \quad (3.3)$$

#### 3.3 Interpolation

Let

$$\pi_{h,1} : [L^2(\mathcal{K}_{h,1})]^d \ni v \mapsto \pi_h v \in [V_{h,k_u,1}]^d \quad (3.4)$$

be a Scott–Zhang type interpolant. Then, for each element  $K \in \mathcal{K}_{h,1}$  we have the following elementwise estimate

$$\|v^e - \pi_h v^e\|_{H^m(K)} \lesssim h^{s-m} \|v^e\|_{H^s(N(K))} \lesssim \|v\|_{H^s(N^l(K))}, \quad m \leq s \leq k_u + 1, \quad m = 0, 1 \quad (3.5)$$

where  $N(K)$  is the set of neighboring elements to  $K$  and  $N^l(K) = p(N(K))$ . In (3.5) the first inequality follows from standard interpolation theory, see [1], and the second from the chain rule in combination with  $L^\infty$  boundedness of derivatives of the closest point map  $p$  in the tubular neighborhood  $U_{\delta_0}(\Gamma)$  which follows from smoothness of  $\Gamma$ .

Next we define the interpolant  $\pi_{h,k_g} : [L^2(\mathcal{K}_h)]^d \rightarrow V_{h,k_u,k_g}^d$  as follows

$$\pi_{h,k_g} v^e|_K = (\pi_{h,1} v^e) \circ G_{K,k_g,1} \quad (3.6)$$

where  $G_{K,k_g,1} = F_{K,1} \circ F_{K,k_g}^{-1} : K_{k_g} \rightarrow K_1$  is a bijection from the curved triangle  $K_{k_g}$  to the corresponding flat triangle  $K_1$ . Using uniform  $L^\infty$  bounds on  $G_{K,k_g,1}$  and its first order derivative we have the estimates

$$\|v^e - \pi_{h,k_g} v^e\|_{H^m(K_{k_g})} \lesssim \|v^e - \pi_{h,1} v^e\|_{H^m(K_1)} \quad (3.7)$$

$$\lesssim h^{s-m} \|v^e\|_{H^s(N(K_1))} \quad (3.8)$$

$$\lesssim h^{s-m} \|v\|_{H^s(N^l(K_1))} \quad (3.9)$$

and thus we conclude that we have the estimate

$$\|v^e - \pi_{h,k_g} v^e\|_{H^m(K_{k_g})} \lesssim h^{s-m} \|v\|_{H^s(N^l(K_1))}, \quad m \leq s \leq k_u + 1, \quad m = 0, 1 \quad (3.10)$$

for all  $K \in \mathcal{K}_{h,k_g}$ . We also have the stability estimate

$$\|\pi_{h,k_g} v^e\|_{H^m(K_{k_g})} \lesssim \|v\|_{H^m(N^l(K_1))}, \quad m = 0, 1 \quad (3.11)$$

When appropriate we simplify the notation and write  $\pi_h = \pi_{h,k_g}$ .

**Lemma 3.1 (Super-approximation)** *For  $v \in V_h$  and  $\chi \in [W_\infty^{k_u+1}(\Gamma)]^3$  is holds*

$$\|\nabla_{\Gamma_h}(I - \pi_{h,k_g})(\chi^e \cdot v)\|_{\Gamma_h} \lesssim h \|\chi\|_{W_\infty^{k_u+1}(\Gamma)} \|v \otimes \nabla_\Gamma\|_{\Gamma_h} \quad (3.12)$$

$$\|\nabla_{\Gamma_h}(I - \pi_{h,k_g})(\chi^e \cdot v)\|_{\Gamma_h} \lesssim \|\chi\|_{W_\infty^{k_u+1}(\Gamma)} \|v\|_{\Gamma_h} \quad (3.13)$$

**Proof.** Let  $I_{h,k_g}$  denote the Lagrange interpolant. Using the fact that  $\pi_{h,k_g}$  is a projection on  $V_{h,k_u,k_g}$  and is  $H^1$  stable (3.11) we have

$$\|(I - \pi_{h,k_g})(\chi^e \cdot v)\|_{H^1(K_{k_g})} \lesssim \|(I - \pi_{h,k_g})(I - I_{h,k_g})(\chi^e \cdot v)\|_{H^1(K_{k_g})} \quad (3.14)$$

$$\lesssim \|(I - I_{h,k_g})(\chi^e \cdot v)\|_{H^1(N(K_1))} \quad (3.15)$$

$$\lesssim h^{k_u} \sum_{K' \in N(K_1)} \|\chi^e \cdot v\|_{H^{k_u+1}(K')} \quad (3.16)$$

$$\lesssim h^{k_u} \sum_{K' \in N(K_1)} \|\chi\|_{[W_\infty^{k_u+1}(K^l)]^3} \|v\|_{H^{k_u+1}(K')} \quad (3.17)$$

where we in the last inequality use the assumption  $\chi \in [W_\infty^{k_u+1}(\Gamma)]^3$ . The proof is finalized by the following estimates

$$h^{k_u} \|v\|_{H^{k_u+1}(K')} = h^{k_u} \|v\|_{H^{k_u}(K')} \lesssim h \|v\|_{H^1(K')} \lesssim \|v\|_{K'} \quad (3.18)$$

where we in the equality use that the  $(k_u + 1)$ :th derivative of a polynomial of order  $k_u$  is zero and we in the inequalities use two inverse estimates yielding (3.12) and (3.13), respectively.  $\square$



### 3.4 Formulation of the Method

The finite element method takes the form: Find  $u_h \in V_h$  such that

$$A_h(u_h, v) = l_h(v) \quad \forall v \in V_h \quad (3.19)$$

The forms are defined by

$$A_h(v, w) = a_h(v, w) + s_h(v, w) \quad (3.20)$$

with

$$a_h(v, w) = (D_{\Gamma_h} v, D_{\Gamma_h} w)_{\Gamma_h} \quad (3.21)$$

$$s_h(v, w) = \beta(h^{-2} n_h \cdot v, n_h \cdot w)_{\Gamma_h} \quad (3.22)$$

$$l_h(v) = (f \circ p, v)_{\Gamma_h} \quad (3.23)$$

where  $\beta > 0$  is a parameter. The form  $s_h$  is added to weakly enforce the tangent condition.

**Remark 3.1** *Depending on available geometry information it may be possible to use other normals than  $n_h$  in the form  $s_h$ . For instance, in surface evolution problems we would typically only have access to a discrete triangulated surface and thus  $n_h$  is a natural choice. In other applications the triangulation may be constructed from a parametrization of the exact surface, for instance a CAD model, and then the exact normal in the nodes is typically available. We elaborate on how a better normal approximation in  $s_h$  would affect the error estimates in Remark 5.1.*

**Remark 3.2** *The finite element method for the symmetric formulation is obtained by replacing  $a_h$  with the form*

$$a_{h, \text{sym}}(v, w) = (\epsilon_{\Gamma_h}(v), \epsilon_{\Gamma_h}(w))_{\Gamma_h} \quad (3.24)$$

## 4 Preliminary Results

### 4.1 Extension and Lifting of Functions

In this section we summarize basic results concerning extension and liftings of functions. We refer to [2] and [3] for further details.

**Extension.** Recalling the definition  $v^e = v \circ p$  of the extension and using the chain rule we obtain the identity

$$\nabla_{\Gamma_h} v^e = B^T \nabla_{\Gamma} v \quad (4.1)$$

where

$$B = P(I - \rho\kappa)P_h : T_x(K) \rightarrow T_{p(x)}(\Gamma) \quad (4.2)$$

and  $\kappa = \nabla \otimes \nabla \rho$  is the curvature tensor (or second fundamental form) which may be expressed in the form

$$\kappa(x) = \sum_{i=1}^2 \frac{\kappa_i^e}{1 + \rho(x)\kappa_i^e} a_i^e \otimes a_i^e \quad (4.3)$$

where  $\kappa_i$  are the principal curvatures with corresponding orthonormal principal curvature vectors  $a_i$ , see [8, Lemma 14.7]. We note that there is  $\delta > 0$  such that the uniform bound

$$\|\kappa\|_{L^\infty(U_\delta(\Gamma))} \lesssim 1 \quad (4.4)$$

holds. Furthermore, we show below that  $B : T_x(K) \rightarrow T_{p(x)}(\Gamma)$  is invertible for  $h \in (0, h_0]$  with  $h_0$  small enough, i.e, there is  $B^{-1} : T_{p(x)}(\Gamma) \rightarrow T_x(K)$  such that

$$BB^{-1} = P, \quad B^{-1}B = P_h \quad (4.5)$$

**Lifting.** The lifting  $w^l$  of a function  $w$  defined on  $\Gamma_h$  to  $\Gamma$  is defined as the push forward

$$(w^l)^e = w^l \circ p = w \quad \text{on } \Gamma_h \quad (4.6)$$

For the derivative it follows that

$$\nabla_{\Gamma_h} w = \nabla_{\Gamma_h} (w^l)^e = B^T \nabla_\Gamma (w^l) \quad (4.7)$$

and thus

$$\nabla_\Gamma (w^l) = B^{-T} \nabla_{\Gamma_h} w \quad (4.8)$$

## 4.2 Extension and Lifting of Vector Valued Functions

We employ componentwise lifting and extension of vector valued functions which directly give the identities:

$$v^e \otimes \nabla_{\Gamma_h} = (v \otimes \nabla_\Gamma) B \quad v \in H^1(\Gamma) \quad (4.9)$$

$$v^l \otimes \nabla_\Gamma = (v \otimes \nabla_{\Gamma_h}) B^{-1} \quad v \in H^1(\Gamma_h) \quad (4.10)$$

## 4.3 Estimates Related to $B$

Using the uniform bound  $\|\kappa\|_{U_{\delta_0}(\Gamma)} \lesssim 1$  and the bound  $\|\rho\|_{L^\infty(\Gamma_h)} \lesssim h^{k_g+1}$  from the geometry approximation assumption it follows that

$$\|B\|_{L^\infty(\Gamma_h)} \lesssim 1, \quad \|B^{-1}\|_{L^\infty(\Gamma)} \lesssim 1 \quad (4.11)$$

$$\|PP_h - B\|_{L^\infty(\Gamma)} \lesssim h^{k_g+1}, \quad \|P_h P - B^{-1}\|_{L^\infty(\Gamma_h)} \lesssim h^{k_g+1} \quad (4.12)$$

For the surface measures on  $\Gamma$  and  $\Gamma_h$  we have the identity

$$d\Gamma = |B| d\Gamma_h \quad (4.13)$$

where  $|B| = |\det(B)|$  is the absolute value of the determinant of  $B$  and we have the following estimates

$$\|1 - |B|\|_{L^\infty(\Gamma_h)} \lesssim h^{k_g+1}, \quad \| |B| \|_{L^\infty(\Gamma_h)} \lesssim 1, \quad \| |B|^{-1} \|_{L^\infty(\Gamma_h)} \lesssim 1 \quad (4.14)$$

**Verification of (4.12).** The first estimate follows directly from (3.1),

$$\|B - PP_h\|_{L^\infty(\Gamma_h)} = \|\rho\kappa P_h\|_{L^\infty(\Gamma_h)} \lesssim h^{k_g+1} \quad (4.15)$$

For the second we first note that for  $\xi \in T_x(K)$  we have the estimate

$$\|\xi\|_{\mathbb{R}^3} \lesssim \|B\xi\|_{\mathbb{R}^3} \quad (4.16)$$

since

$$\|B\xi\|_{\mathbb{R}^3} = \|P(I - \rho\kappa)PP_h\xi\|_{\mathbb{R}^3} \quad (4.17)$$

$$\gtrsim \|PP_h\xi\|_{\mathbb{R}^3} - h^{k_g+1}\|P_h\xi\|_{\mathbb{R}^3} \quad (4.18)$$

$$= \|(I - Q)P_h\xi\|_{\mathbb{R}^3} - h^{k_g+1}\|P_h\xi\|_{\mathbb{R}^3} \quad (4.19)$$

$$\gtrsim \|P_h\xi\|_{\mathbb{R}^3} - h^{k_g}\|P_h\xi\|_{\mathbb{R}^3} - h^{k_g+1}\|P_h\xi\|_{\mathbb{R}^3} \quad (4.20)$$

$$\gtrsim \|\xi\|_{\mathbb{R}^3} \quad (4.21)$$

for  $h \in (0, h_0]$  with  $h_0$  small enough. Thus it follows from (4.16) that  $B$  is invertible and for  $\eta \in T_{p(x)}(\Gamma)$  we have the estimate

$$\|(B^{-1} - P_hP)\eta\|_{\mathbb{R}^3} \lesssim \|B(B^{-1} - P_hP)\eta\|_{\mathbb{R}^3} \quad (4.22)$$

$$\lesssim \|(P - BP_hP)\eta\|_{\mathbb{R}^3} \quad (4.23)$$

$$= \|(P - B)\eta\|_{\mathbb{R}^3} \quad (4.24)$$

$$\lesssim \|(P - PP_h)\eta\|_{\mathbb{R}^3} + \|(PP_h - B)\eta\|_{\mathbb{R}^3} \quad (4.25)$$

$$\lesssim \|PQ_hP\eta\|_{\mathbb{R}^3} + \|(PP_h - B)\eta\|_{\mathbb{R}^3} \quad (4.26)$$

$$\lesssim (h^{2k_g} + h^{k_g+1})\|\eta\|_{\mathbb{R}^3} \quad (4.27)$$

where we first used (4.16) and then (3.1) and (4.15). It thus follows that, in the operator norm,

$$\|(B^{-1} - P_hP)\|_{\mathbb{R}^3} = \sup_{\eta \in T_{p(x)}(\Gamma)} \frac{\|(B^{-1} - P_hP)\eta\|_{\mathbb{R}^3}}{\|\eta\|_{\mathbb{R}^3}} \lesssim h^{2k_g} + h^{k_g+1} \lesssim h^{k_g+1} \quad (4.28)$$

for  $k_g \geq 1$ .

## 4.4 Norm Equivalences

In order to conveniently deal with extensions and liftings we will write  $v = v^e$  and  $v = v^l$  when there is no risk for confusion. In this way we may think of functions as being defined both on  $\Gamma$  and  $\Gamma_h$  and we can form the sum of function spaces on  $\Gamma$  and  $\Gamma_h$ , for instance,  $L^2(\Gamma) + L^2(\Gamma_h)$  or  $V + V_h$ . In view of the bounds in Section 4.3 and the identities (4.1) and (4.8) we obtain the following equivalences for  $v \in H^1(\Gamma) + H^1(\Gamma_h)$

$$\|v\|_{L^2(\Gamma)} \sim \|v\|_{L^2(\Gamma_h)} \quad \text{and} \quad \|\nabla_\Gamma v\|_{L^2(\Gamma)} \sim \|\nabla_{\Gamma_h} v\|_{L^2(\Gamma_h)} \quad (4.29)$$

## 5 Error Estimates

### 5.1 Norms

For a continuous semidefinite form  $\alpha$  on a Hilbert space  $H$  we let  $\|v\|_\alpha^2 = \alpha(v, v)$  be the seminorm associated with  $\alpha$  on  $H$ . We also use the standard notation

$$\|v\|_h^2 = \|v\|_{A_h}^2 = \|v\|_{a_h}^2 + \|v\|_{s_h}^2 = \|D_{\Gamma_h} v\|_{\Gamma_h}^2 + h^{-2} \|n_h \cdot v\|_{\Gamma_h}^2 \quad (5.1)$$

for the discrete energy norm.

### 5.2 Coercivity and Continuity

**Lemma 5.1** *It holds*

$$\|v\|_h^2 \lesssim A_h(v, v) \quad v \in V + V_h \quad (5.2)$$

and

$$|A_h(v, w)| \lesssim \|v\|_h \|w\|_h \quad v, w \in V + V_h \quad (5.3)$$

**Proof.** The first inequality holds by definition since  $\|v\|_h = \|v\|_{A_h}$ . The second inequality directly follows by the Cauchy–Schwarz inequality since  $A_h(\cdot, \cdot)$  is an inner product.  $\square$

### 5.3 Basic Lemmas

**Lemma 5.2** *If  $v \in H_{\text{tan}}^1(\Gamma)$  satisfies  $D_\Gamma v = 0$  then  $v = 0$ .*

**Proof.** The Riemannian curvature tensor, see [4], is the mapping

$$R(a, b, v) = D_a D_b v - D_b D_a v - D_{[a, b]} v \quad (5.4)$$

where  $D_a v = D_\Gamma v \cdot a = P(a \cdot \nabla) v$  is the covariant derivative in the direction of the tangential vector field  $a$  and  $[a, b]$  is the tangent vector field given by the Lie bracket

$$[a, b] = \partial_b a - \partial_a b \quad (5.5)$$

where we recall that  $\partial_b a = (u^e \otimes \nabla) \cdot b$ , see (2.3). All derivatives in (5.4) cancel so that  $Rv$  is a tangential vector field which does not depend on any derivatives of  $v$ . In the case of an embedded codimension one surface in  $\mathbb{R}^3$  we have the identity

$$R(a, b, v) = (b \cdot \kappa \cdot v) \kappa \cdot a - (a \cdot \kappa \cdot v) \kappa \cdot b \quad (5.6)$$

where  $\kappa$  is the curvature tensor (4.3) of  $\Gamma$ . See below for a verification.

Next let  $\{t_i\}_{i=1}^2$  be a smooth orthonormal basis to  $T_x(\Gamma)$  in the vicinity of a point  $x \in \Gamma$ , i.e., all tangential vector fields can be written as a linear combination  $v = \sum_{i=1}^2 v_i t_i$  with coordinate functions  $v_i$ . We then have the identity

$$R(t_2, t_1, t_1) \cdot t_2 = R(t_1, t_2, t_2) \cdot t_1 = K \quad (5.7)$$

where  $K = \kappa_1 \kappa_2$  is the Gauss curvature and it also holds

$$R(t_2, t_1, t_2) \cdot t_2 = R(t_1, t_2, t_1) \cdot t_1 = 0 \quad (5.8)$$

See below for verification.

If  $D_\Gamma v = 0$  we have  $D_a v = 0$  for all tangential vector fields  $a$  and thus we can conclude that  $R(a, b, v) = 0$  for all tangential vector fields  $a, b$ . Expanding  $v$  in the orthonormal frame we also have the identity

$$0 = R(a, b, v) \cdot w = R(a, b, \sum_{i=1}^2 v_i t_i) \cdot w = \sum_{i=1}^2 v_i R(a, b, t_i) \cdot w \quad (5.9)$$

Setting  $a = t_1$ ,  $b = t_2$ , and  $w = t_1$  we get

$$0 = v_2 K \quad (5.10)$$

and setting  $a = t_2$ ,  $b = t_1$ , and  $w = t_2$  we get

$$0 = v_1 K \quad (5.11)$$

We can therefore conclude that in a point with nonzero Gauss curvature a covariantly constant vector field must be zero.

For any closed compact smooth surface embedded in  $\mathbb{R}^3$  there is at least one point  $x \in \Gamma$  where  $K \neq 0$ , see [15, Theorem 4, p. 88]. Furthermore, parallel transport implies that the pointwise norm of the vector field is constant along geodesic curves. Using the fact that a closed compact manifold is geodesically complete in the sense that each point  $y \in \Gamma$  is connected to  $x$  by a geodesic curve we can conclude that  $v(y) = 0$ , since  $v(x) = 0$  and  $v(y)$  is a parallel transport of  $v(x)$ . Finally, using the fact that smooth vector fields are dense in  $H_{\text{tan}}^1(\Gamma)$  the desired result follows.

**Verification of (5.6).** Recall the directional and covariant derivatives introduced in Section 2.2, i.e.

$$\partial_a v = (v \otimes \nabla_\Gamma) \cdot a, \quad D_a v = P \partial_a v \quad (5.12)$$

for a tangential vector field  $a$ . We then have

$$D_a D_b v = P \partial_a P \partial_b v \quad (5.13)$$

$$= P \partial_a (\partial_b v - (n \cdot \partial_b v) n) \quad (5.14)$$

$$= P \partial_a \partial_b v - P (\partial_a (n \cdot \partial_b v) n + (n \cdot \partial_b v) \partial_a n) \quad (5.15)$$

$$= P \partial_b \partial_a v - (n \cdot \partial_b v) \kappa \cdot a \quad (5.16)$$

$$= P \partial_a \partial_b v + (b \cdot \kappa \cdot v) \kappa \cdot a \quad (5.17)$$

Here we used that identities

$$Pn = 0, \quad P\kappa = \kappa, \quad \partial_a n = \kappa \cdot a, \quad n \cdot \partial_b v = -b \cdot \kappa \cdot v \quad (5.18)$$

where the last formula follows from the fact that  $v \cdot n = 0$ , which leads to

$$0 = \partial_b(n \cdot v) = n \cdot (\partial_b v) + \partial_b n \cdot v = n \cdot (\partial_b v) + b \cdot \kappa \cdot v \quad (5.19)$$

We thus obtain

$$R(a, b, v) = D_a D_b v - D_b D_a v - D_{[a, b]} v \quad (5.20)$$

$$= \underbrace{P(\partial_a \partial_b v - \partial_b \partial_a v - \partial_{[a, b]} v)}_{\star} \quad (5.21)$$

$$+ (a \cdot \kappa \cdot v) \kappa \cdot b - (b \cdot \kappa \cdot v) \kappa \cdot a \\ = (a \cdot \kappa \cdot v) \kappa \cdot b - (b \cdot \kappa \cdot v) \kappa \cdot a \quad (5.22)$$

Here we used the identity

$$\partial_a \partial_b v_i = \partial_b \partial_a v_i + \partial_{\partial_a b} v_i \quad (5.23)$$

for each component  $v_i$  in  $v$ , to conclude that

$$\partial_a \partial_b v - \partial_b \partial_a v = \partial_{\partial_a b} v - \partial_{\partial_b a} v = \partial_{[a, b]} v \quad (5.24)$$

and thus  $\star = 0$ . This concludes the verification of (5.6).

**Verification of (5.7) and (5.8).** We directly obtain

$$R(t_2, t_1, t_1) \cdot t_2 = (t_1 \cdot \kappa \cdot t_1)(t_2 \cdot \kappa \cdot t_2) - (t_2 \cdot \kappa \cdot t_1)(t_2 \cdot \kappa \cdot t_2) = \det(\kappa) = \kappa_1 \kappa_2 \quad (5.25)$$

where  $\det(\kappa)$  is the determinant of the  $2 \times 2$  tangential part of  $\kappa$  and we used the fact that  $T = [t_1 \ t_2]$  is orthogonal and  $\det(T^T \kappa T) = \det \kappa$ . For the second identity we readily get

$$R(t_1, t_2, t_1) \cdot t_1 = (t_2 \cdot \kappa \cdot t_1)(t_1 \cdot \kappa \cdot t_1) - (t_1 \cdot \kappa \cdot t_1)(t_2 \cdot \kappa \cdot t_1) = 0 \quad (5.26)$$

and we get the corresponding results for both verifications if we switch  $t_1$  and  $t_2$ .  $\square$

**Lemma 5.3 (Poincaré Inequality)** *For all  $v \in H_{\text{tan}}^1(\Gamma)$  there is a constant such that*

$$\|v\|_{\Gamma} \lesssim \|D_{\Gamma} v\|_{\Gamma} \quad (5.27)$$

**Proof.** Assume that (5.27) does not hold. Then there is a sequence  $\{v_k\}_{k=1}^{\infty}$  in  $H_{\text{tan}}^1(\Gamma)$  such that

$$\|v_k\|_{\Gamma} \geq k \|D_{\Gamma} v_k\|_{\Gamma} \quad (5.28)$$

Setting  $w_k = v_k / \|v_k\|_\Gamma$  we obtain

$$\|D_\Gamma w_k\|_\Gamma \leq k^{-1} \quad (5.29)$$

and therefore  $\{w_k\}_{k=1}^\infty$  is bounded in  $H_{\tan}^1(\Gamma)$ . Using Rellich's compactness theorem, see [7], there is a subsequence  $\{w_{k_j}\}_{j=1}^\infty$  and a tangential vector field  $w \in L^2(\Gamma)$  such that

$$w_{k_j} \rightarrow w \quad \text{in } L^2(\Gamma) \quad (5.30)$$

Then  $\|w\|_\Gamma = 1$  and  $\|D_\Gamma w\|_\Gamma = 0$  but this is a contradiction in view of Lemma 5.2.  $\square$

**Lemma 5.4** *For all tangential vector fields  $v_t \in H_{\tan}^m(\Gamma)$ , and  $m = 1, 2, \dots$  there is a constant such that*

$$\|v_t \otimes \nabla_\Gamma^m\|_\Gamma \lesssim \sum_{k=1}^m \|(D_\Gamma)^k v_t\|_\Gamma \quad (5.31)$$

and as a consequence

$$\|v_t\|_{H^m(\Gamma)} \lesssim \|v_t\|_{H_{\tan}^m(\Gamma)} \quad (5.32)$$

**Proof.** Let  $\{t_i\}_{i=1}^2$  be a smoothly varying orthonormal basis on  $\Gamma$  and let  $\{e_j\}_{j=1}^3$  be the Cartesian coordinate system in  $\mathbb{R}^3$ . Then we have the identity

$$t_i = \sum_{j=1}^3 \alpha_{ij} e_j \quad (5.33)$$

for some smooth functions  $\alpha_{ij} : \Gamma \rightarrow \mathbb{R}$ . With respect to  $\{t_i\}_{i=1}^2$  we can express a  $(n, 0)$  tangential tensor as

$$X = \sum_{i_1, \dots, i_n=1}^2 \underbrace{X_{i_1 i_2 \dots i_n}}_{X_{\mathcal{I}}} \underbrace{t_{i_1} \otimes t_{i_2} \otimes \dots \otimes t_{i_n}}_{T_{\mathcal{I}}} \quad (5.34)$$

$$= \sum_{i_1, \dots, i_n=1}^2 \sum_{j_1, \dots, j_n=1}^3 X_{i_1 i_2 \dots i_n} \underbrace{\alpha_{i_1 j_1} \alpha_{i_2 j_2} \dots \alpha_{i_n j_n}}_{\alpha_{\mathcal{I}\mathcal{J}}} \underbrace{e_{j_1} \otimes \dots \otimes e_{j_n}}_{E_{\mathcal{J}}} \quad (5.35)$$

$$= \sum_{\mathcal{I} \leq 2} \sum_{\mathcal{J} \leq 3} X_{\mathcal{I}} \alpha_{\mathcal{I}\mathcal{J}} E_{\mathcal{J}} \quad (5.36)$$

where  $X_{\mathcal{I}} = X_{i_1 i_2 \dots i_n}$  are the coefficients of  $X_t$ . Let Recall the notation in (5.12) for a derivative  $\partial_a$  and covariant derivative  $D_a$  along a tangential vector field  $a$ . Taking the

derivative of  $X$  along  $a$  gives

$$\partial_a X = \sum_{\mathcal{I} \leq 2} \sum_{\mathcal{J} \leq 3} (\partial_a X_{\mathcal{I}}) \alpha_{\mathcal{I}\mathcal{J}} E_{\mathcal{J}} + X_{\mathcal{I}} (\partial_a \alpha_{\mathcal{I}\mathcal{J}}) E_{\mathcal{J}} \quad (5.37)$$

$$= \sum_{\mathcal{I} \leq 2} \sum_{\mathcal{J} \leq 3} (\partial_a X_{\mathcal{I}}) T_{\mathcal{I}} + X_{\mathcal{I}} (\partial_a \alpha_{\mathcal{I}\mathcal{J}}) E_{\mathcal{J}} \quad (5.38)$$

$$= \sum_{\mathcal{I} \leq 2} \sum_{\mathcal{J} \leq 3} (\partial_a X_{\mathcal{I}}) T_{\mathcal{I}} + \text{proj}_{\Gamma}(X_{\mathcal{I}} (\partial_a \alpha_{\mathcal{I}\mathcal{J}}) E_{\mathcal{J}}) \quad (5.39)$$

$$\begin{aligned} &+ (X_{\mathcal{I}} (\partial_a \alpha_{\mathcal{I}\mathcal{J}}) E_{\mathcal{J}} - \text{proj}_{\Gamma}(X_{\mathcal{I}} (\partial_a \alpha_{\mathcal{I}\mathcal{J}}) E_{\mathcal{J}})) \\ &= \sum_{\mathcal{I} \leq 2} \sum_{\mathcal{J} \leq 3} D_a X_t + (T_{\mathcal{I}} : X) (\partial_a \alpha_{\mathcal{I}\mathcal{J}}) E_{\mathcal{J}} - \text{proj}_{\Gamma}((T_{\mathcal{I}} : X) (\partial_a \alpha_{\mathcal{I}\mathcal{J}}) E_{\mathcal{J}}) \end{aligned} \quad (5.40)$$

where we in the last equality for the first term use that the covariant derivative is the projected part of the tangential derivative and for the remaining terms we use that the coordinates of  $X$  can be expressed  $X_{\mathcal{I}} = T_{\mathcal{I}} : X$  where ":" denotes the contraction in each tensorial dimension as  $t_1, t_2$  are orthonormal tangent vectors. Thus, the derivative of a tangential tensor  $X$  can be expressed as a linear combination of the covariant derivative  $D_{\Gamma} X_t$  and  $X_t$  itself.

Iterating this result on a tangent vector field  $v_t$  yields

$$\|v_t \otimes \nabla_{\Gamma}^m\|_{\Gamma} \lesssim \sum_{j=0}^m \|(D_{\Gamma})^j v_t\|_{\Gamma} \quad (5.41)$$

and (5.31) follows by applying Lemma 5.3.  $\square$

**Lemma 5.5 (Discrete Poincaré Inequality)** *For  $k_g \geq 1$  and all  $v \in V_h$  and  $h \in (0, h_0]$ , with  $h_0$  small enough, there is a constant such that*

$$\|v\|_{\Gamma_h} \lesssim \|v\|_h \quad (5.42)$$

**Proof.** Using norm equivalence (4.29), splitting in tangent and normal components and the triangle inequality we have

$$\|v\|_{\Gamma_h} \lesssim \|v\|_{\Gamma} \lesssim \|v_t\|_{\Gamma} + \|v_n\|_{\Gamma} \quad (5.43)$$

The tangent component can be estimated using the Poincaré inequality, Lemma 5.3, on  $\Gamma$

$$\|v_t\|_{\Gamma} \lesssim \|P v_t \otimes \nabla_{\Gamma}\|_{\Gamma} \quad (5.44)$$

$$\lesssim \|P v \otimes \nabla_{\Gamma}\|_{\Gamma} + \|P(v_n n) \otimes \nabla_{\Gamma}\|_{\Gamma} \quad (5.45)$$

$$\lesssim \|P v \otimes \nabla_{\Gamma_h}\|_{\Gamma_h} + \|v_n\|_{\Gamma} \quad (5.46)$$

$$\lesssim \|P_h v \otimes \nabla_{\Gamma_h}\|_{\Gamma_h} + \|(P - P_h) v \otimes \nabla_{\Gamma_h}\|_{\Gamma_h} + \|v_n\|_{\Gamma} \quad (5.47)$$

$$\lesssim \|P_h v \otimes \nabla_{\Gamma_h}\|_{\Gamma_h} + h^{k_g} \|v \otimes \nabla_{\Gamma_h}\|_{\Gamma_h} + \|v_n\|_{\Gamma} \quad (5.48)$$

$$\lesssim \|P_h v \otimes \nabla_{\Gamma_h}\|_{\Gamma_h} + h^{k_g-1} \|v\|_{\Gamma_h} + \|v_n\|_{\Gamma} \quad (5.49)$$



where we used the identity  $P(v_n n) \otimes \nabla_\Gamma = v_n \kappa$  and the uniform bound  $\|\kappa\|_{L^\infty(\Gamma)} \lesssim 1$ , changed domain of integration in the first term, added and subtracted  $P_h$  and used the triangle inequality, and finally we used an inverse inequality. For the normal component we have

$$\|v_n\|_\Gamma \lesssim \|(n - n_h) \cdot v\|_\Gamma + \|n_h \cdot v\|_\Gamma \quad (5.50)$$

$$\lesssim h^{k_g} \|v\|_\Gamma + \|v_{n_h}\|_\Gamma \quad (5.51)$$

$$\lesssim h^{k_g} \|v\|_{\Gamma_h} + \|v_{n_h}\|_{\Gamma_h} \quad (5.52)$$

Thus we conclude that

$$\|v\|_\Gamma \lesssim \|P_h v \otimes \nabla_{\Gamma_h}\|_{\Gamma_h} + h^{k_g-1}(1+h)\|v\|_{\Gamma_h} + \|v_{n_h}\|_{\Gamma_h} \quad (5.53)$$

$$\lesssim \|v\|_h + h^{k_g-1}\|v\|_{\Gamma_h} \quad (5.54)$$

Finally, using norm equivalence (4.29) and a kickback argument this concludes the proof for all  $h \in (0, h_0]$  with  $h_0$  small enough.  $\square$

**Lemma 5.6** *For  $k_g \geq 1$  and all  $v \in V_h$ ,  $h \in (0, h_0]$  with  $h_0$  small enough, there is a constant such that*

$$\|v \otimes \nabla_{\Gamma_h}\|_{\Gamma_h} \lesssim \|v\|_h \quad (5.55)$$

**Proof.** Starting with the estimate

$$\|v \otimes \nabla_{\Gamma_h}\|_{\Gamma_h} \leq \|P_h v \otimes \nabla_{\Gamma_h}\|_{\Gamma_h} + \|Q_h v \otimes \nabla_{\Gamma_h}\|_{\Gamma_h} \quad (5.56)$$

$$\leq \|P_h v \otimes \nabla_{\Gamma_h}\|_{\Gamma_h} + \underbrace{\|Q_h v_t \otimes \nabla_{\Gamma_h}\|_{\Gamma_h}}_I + \underbrace{\|Q_h(v_n n) \otimes \nabla_{\Gamma_h}\|_{\Gamma_h}}_{II} \quad (5.57)$$

where we used the orthogonal decomposition

$$v = (Pv) + (Qv) = v_t + (v \cdot n)n = v_t + v_n n \quad (5.58)$$

**Term I.** Let  $\{t_i\}_{i=1}^2$  be an orthonormal tangential coordinate system on  $\Gamma$  and let  $\{e_j\}_{j=1}^3$  be the Cartesian coordinate system in  $\mathbb{R}^3$ . Then we have the identity

$$v_t = \sum_{i=1}^2 v_i t_i = \sum_{i=1}^2 \sum_{j=1}^3 v_i \alpha_{ij} e_j \quad (5.59)$$

for some smooth functions  $\alpha_{ij} : \Gamma \rightarrow \mathbb{R}$ . Taking the derivative we obtain

$$v_t \otimes \nabla_{\Gamma_h} = \sum_{i=1}^2 \sum_{j=1}^3 e_j \otimes \nabla_{\Gamma_h}(v_i \alpha_{ij}) \quad (5.60)$$

$$= \sum_{i=1}^2 \sum_{j=1}^3 \alpha_{ij} e_j \otimes (\nabla_{\Gamma_h} v_i) + \sum_{i=1}^2 \sum_{j=1}^3 v_i e_j \otimes (\nabla_{\Gamma_h} \alpha_{ij}) \quad (5.61)$$

$$= \sum_{i=1}^2 t_i \otimes (\nabla_{\Gamma_h} v_i) + \sum_{i=1}^2 \sum_{j=1}^3 v_i e_j \otimes (\nabla_{\Gamma_h} \alpha_{ij}) \quad (5.62)$$

Thus we conclude that

$$Q_h v_t \otimes \nabla_{\Gamma_h} = \sum_{i=1}^2 \sum_{j=1}^3 Q_h t_i \otimes (\nabla_{\Gamma_h} v_i) + \sum_{i=1}^2 \sum_{j=1}^3 v_i (Q_h e_j) \otimes (\nabla_{\Gamma_h} \alpha_{ij}) \quad (5.63)$$

and by the bounds  $\|P \cdot n_h\|_{L^\infty(\Gamma_h)} \lesssim h^{k_g}$  and  $\|\nabla_{\Gamma_h} \alpha_{ij}\|_{L^\infty(\Gamma_h)} \lesssim 1$  we have

$$\|Q_h v_t \otimes \nabla_{\Gamma_h}\|_{\Gamma_h} \lesssim \sum_{i=1}^2 h^{k_g} \|\nabla_{\Gamma_h} v_i\|_{\Gamma_h} + \|v_i\|_{\Gamma_h} = \star \quad (5.64)$$

Next, using the identity  $v_i = v \cdot t_i$ , which holds since  $\{t_1, t_2, n\}$  is an orthonormal coordinate system, we may add and subtract an interpolant and then use super-approximation (3.13) and an inverse inequality as follows

$$\|\nabla_{\Gamma_h} v_i\|_{\Gamma_h} \leq \|\nabla_{\Gamma_h} (I - \pi_h)(v \cdot t_i)\|_{\mathcal{K}_h} + \|\nabla_{\Gamma_h} \pi_h v_i\|_{\mathcal{K}_h} \quad (5.65)$$

$$\leq \|v\|_{\mathcal{K}_h} + h^{-1} \|\pi_h v_i\|_{\mathcal{K}_h} \quad (5.66)$$

$$\leq \|v\|_{\Gamma_h} + h^{-1} \|v_i\|_{\Gamma_h} \quad (5.67)$$

$$\leq (1 + h^{-1}) \|v\|_{\Gamma_h} \quad (5.68)$$

$$\leq (1 + h^{-1}) (\|v_{t_h}\|_{\Gamma_h} + \|v_{n_h}\|_{\Gamma_h}) \quad (5.69)$$

where we used the  $L^2$  stability of  $\pi_h$  and the trivial estimate  $|t_i| = |v \cdot t_i| \leq |v|$ . We also have the estimate

$$\|v_i\|_{\Gamma_h} \leq \|v\|_{\Gamma_h} \quad (5.70)$$

$$\lesssim \|v_{t_h}\|_{\Gamma_h} + \|v_{n_h}\|_{\Gamma_h} \quad (5.71)$$

Thus we obtain

$$\star \lesssim (1 + h^{k_g-1}) (\|v_{t_h}\|_{\Gamma_h} + \|v_{n_h}\|_{\Gamma_h}) \quad (5.72)$$

$$\lesssim (1 + h^{k_g-1}) (\|D_{\Gamma_h} v\|_{\Gamma_h} + \|v_{n_h}\|_{\Gamma_h}) \quad (5.73)$$

$$\lesssim \|v\|_h \quad (5.74)$$

for  $k_g \geq 1$ . Here we used the estimate

$$\|v_{t_h}\|_{\Gamma_h} \lesssim \|D_{\Gamma_h} v\|_{\Gamma_h} + \|v_{n_h}\|_{\Gamma_h} \quad (5.75)$$

which follows by using the Poincaré inequality

$$\|v_{t_h}\|_{\Gamma_h} \lesssim \|D_{\Gamma_h} v_{t_h}\|_{\Gamma_h} \quad (5.76)$$

$$\lesssim \|D_{\Gamma_h} v\|_{\Gamma_h} + \|D_{\Gamma_h} (v_{n_h} n_h)\|_{\Gamma_h} \quad (5.77)$$

$$\lesssim \|D_{\Gamma_h} v\|_{\Gamma_h} + \|v_{n_h}\|_{\Gamma_h} \quad (5.78)$$

where at last we used the identity

$$D_{\Gamma_h} (v_{n_h} n_h) = P_h ((v_{n_h} n_h) \otimes \nabla) P_h = P_h (v_{n_h} \kappa_h + n_h \otimes \nabla_{\Gamma_h} v_{n_h}) P_h = v_{n_h} \kappa_h \quad (5.79)$$

with  $\kappa_h = n_h \otimes \nabla$  the discrete curvature tensor and we used the fact that  $\kappa_h$  is a tangential tensor. By the uniform bound  $\|\kappa_h\|_{L^\infty(\Gamma_h)} \lesssim 1$  we conclude that  $\|D_{\Gamma_h} (v_{n_h} n_h)\|_{\Gamma_h} \lesssim \|v_{n_h}\|_{\Gamma_h}$ .

**Term II.** Proceeding in the same way as above

$$\|Q_h(v_n n) \otimes \nabla_{\Gamma_h}\|_{\Gamma_h} \leq \|\nabla_{\Gamma_h} v_n\|_{\Gamma_h} \quad (5.80)$$

$$\leq \|\nabla_{\Gamma_h}(I - \pi_h)v_n\|_{\Gamma_h} + \|\nabla_{\Gamma_h}(\pi_h v_n)\|_{\Gamma_h} \quad (5.81)$$

$$\lesssim h\|v \otimes \nabla_{\Gamma_h}\|_{\Gamma_h} + h^{-1}\|\pi_h v_n\|_{\Gamma_h} \quad (5.82)$$

$$\lesssim h\|v \otimes \nabla_{\Gamma_h}\|_{\Gamma_h} + h^{-1}\|v_n\|_{\Gamma_h} \quad (5.83)$$

where we used super-approximation (3.12), an inverse inequality, and the  $L^2$  stability of  $\pi_h$ . The first term on the right hand side can be hidden using a kick back argument. In the second term we need to replace  $n$  by  $n_h$ , we proceed as follows

$$\|v_n\|_{\Gamma_h} \leq \|v_{n_h}\|_{\Gamma_h} + h^{k_g}\|v\|_{\Gamma_h} \quad (5.84)$$

$$\lesssim \|v_{n_h}\|_{\Gamma_h} + h^{k_g}(\|v_{t_h}\|_{\Gamma_h} + \|v_{t_h}\|_{\Gamma_h}) \quad (5.85)$$

$$\lesssim (1 + h^{k_g})\|v_{n_h}\|_{\Gamma_h} + h^{k_g}(\|D_{\Gamma_h} v\|_{\Gamma_h} + \|v_{n_h}\|_{\Gamma_h}) \quad (5.86)$$

$$\lesssim (1 + h^{k_g})\|v_{n_h}\|_{\Gamma_h} + h^{k_g}\|D_{\Gamma_h} v\|_{\Gamma_h} \quad (5.87)$$

Thus we have

$$h^{-1}\|v_n\|_{\Gamma_h} \lesssim (h^{-1} + h^{k_g-1})\|v_{n_h}\|_{\Gamma_h} + h^{k_g-1}\|D_{\Gamma_h} v\|_{\Gamma_h} \lesssim \|v\|_h \quad (5.88)$$

for  $k_g \geq 1$ . □

## 5.4 Interpolation

For  $v \in H_{\tan}^{k_u+1}(\Gamma)$  we have the following interpolation error estimate in the energy norm

$$\|v - \pi_h v\|_h \lesssim h^{k_u}\|v\|_{H_{\tan}^{k_u+1}(\Gamma)} \quad (5.89)$$

To verify (5.89) we note that

$$\|v - \pi_h v\|_h^2 = \|D_{\Gamma}(v - \pi_h v)\|_{\Gamma_h}^2 + h^{-2}\|n_h \cdot (v - \pi_h v)\|_{\Gamma_h}^2 \quad (5.90)$$

$$\leq \|(v - \pi_h v) \otimes \nabla_{\Gamma_h}\|_{\Gamma_h}^2 + h^{-2}\|(v - \pi_h v)\|_{\Gamma_h}^2 \quad (5.91)$$

$$\lesssim h^{k_u}\|v\|_{H^{k_u+1}(\Gamma)} \quad (5.92)$$

$$\lesssim h^{k_u}\|v\|_{H_{\tan}^{k_u+1}(\Gamma)} \quad (5.93)$$

where we used the interpolation error estimate (3.10) and at last Lemma 5.4 to pass to the Sobolev norm based on covariant derivatives.

## 5.5 Estimates of Geometric Errors

Define the geometry error forms

$$Q_a(v, w) = a(v, w) - a_h(v, w), \quad Q_l(v) = l(v) - l_h(v) \quad (5.94)$$

Before proceeding with the estimates we formulate a useful lemma

**Lemma 5.7** For  $h \in (0, h_0]$  with  $h_0$  small enough, and  $v \in H^1(\Gamma_h)$  there are constants such that

$$\|(D_\Gamma v^l)^e - D_{\Gamma_h} v\|_{\Gamma_h} \lesssim h^{k_g} \|v \otimes \nabla_{\Gamma_h}\|_{\Gamma_h} \quad (5.95)$$

**Proof.** Using the identity (4.10) and adding and subtracting suitable terms we obtain

$$(D_\Gamma v^l)^e - D_{\Gamma_h} v = (Pv^l \otimes \nabla_\Gamma)^e - P_h v \otimes \nabla_{\Gamma_h} \quad (5.96)$$

$$= Pv \otimes \nabla_{\Gamma_h} B^{-1} - P_h v \otimes \nabla_{\Gamma_h} \quad (5.97)$$

$$= (P - P_h)v \otimes \nabla_{\Gamma_h} B^{-1} \quad (5.98)$$

$$+ P_h v \otimes \nabla_{\Gamma_h} (B^{-1} - P)$$

$$+ P_h v \otimes \nabla_{\Gamma_h} (P - P_h)$$

By the triangle inequality we then have the estimate

$$\|(D_\Gamma v^l)^e - D_{\Gamma_h} v\|_{\Gamma_h} \lesssim \|P - P_h\|_{L^\infty(\Gamma_h)} \|v \otimes \nabla_{\Gamma_h}\|_{\Gamma_h} \|B^{-1}\|_{L^\infty(\Gamma_h)} \quad (5.99)$$

$$+ \|P_h\|_{L^\infty(\Gamma_h)} \|v \otimes \nabla_{\Gamma_h}\|_{\Gamma_h} \|B^{-1} - P\|_{L^\infty(\Gamma_h)}$$

$$+ \|P_h\|_{L^\infty(\Gamma_h)} \|v \otimes \nabla_{\Gamma_h}\|_{\Gamma_h} \|P - P_h\|_{L^\infty(\Gamma_h)}$$

$$\lesssim h^{k_g} \|v \otimes \nabla_{\Gamma_h}\|_{\Gamma_h} \quad (5.100)$$

where we used the bounds (4.12), (4.14), and  $\|P - P_h\|_{L^\infty(\Gamma_h)} \lesssim h^{k_g}$ .  $\square$

**Lemma 5.8 (Geometric Errors)** For  $v, w \in V + V_h$ ,  $h \in (0, h_0]$  with  $h_0$  small enough, there are constants such that

$$Q_a(v, w) \lesssim h^{k_g} \|v\|_{H^1(\Gamma_h)} \|w\|_{H^1(\Gamma_h)} \quad (5.101)$$

$$Q_l(w) \lesssim h^{k_g+1} \|f\|_\Gamma \|w\|_{H^1(\Gamma_h)} \quad (5.102)$$

**Proof. Estimate (5.101).** Changing domain of integration from  $\Gamma$  to  $\Gamma_h$  in the first term and adding and subtracting suitable terms we obtain

$$Q_a(v, w) = (D_\Gamma v^l, D_\Gamma w^l)_\Gamma - (D_{\Gamma_h} v, D_{\Gamma_h} w)_{\Gamma_h} \quad (5.103)$$

$$= ((D_\Gamma v^l)^e, (D_\Gamma w^l)^e |B|)_{\Gamma_h} - (D_{\Gamma_h} v, D_{\Gamma_h} w)_{\Gamma_h} \quad (5.104)$$

$$= (D_\Gamma v^l, D_\Gamma w^l (|B| - 1))_{\Gamma_h} \quad (5.105)$$

$$+ (D_\Gamma v^l, D_\Gamma w^l)_{\Gamma_h} - (D_{\Gamma_h} v, D_{\Gamma_h} w)_{\Gamma_h}$$

Here the first term on the right hand is directly estimated using (4.14),

$$(D_\Gamma v^l, D_\Gamma w^l (|B| - 1))_{\Gamma_h} \leq \|D_\Gamma v^l\|_{\Gamma_h} \|D_\Gamma w^l\|_{\Gamma_h} \|1 - |B|\|_{L^\infty(\Gamma_h)} \quad (5.106)$$

$$\lesssim h^{k_g+1} \|v \otimes \nabla_{\Gamma_h}\|_{\Gamma_h} \|w \otimes \nabla_{\Gamma_h}\|_{\Gamma_h} \quad (5.107)$$

For the second term we add and subtract suitable terms and employ (5.95),

$$(D_\Gamma v^l, D_\Gamma w^l)_{\Gamma_h} - (D_{\Gamma_h} v, D_{\Gamma_h} w)_{\Gamma_h} = (D_\Gamma v^l - D_{\Gamma_h} v, D_\Gamma w^l)_{\Gamma_h} + (D_{\Gamma_h} v, D_\Gamma w^l - D_{\Gamma_h} w)_{\Gamma_h} \quad (5.108)$$

$$\leq \|D_\Gamma v^l - D_{\Gamma_h} v\|_{\Gamma_h} \|D_\Gamma w^l\|_{\Gamma_h} + \|D_{\Gamma_h} v\|_{\Gamma_h} \|D_\Gamma w^l - D_{\Gamma_h} w\|_{\Gamma_h} \quad (5.109)$$

$$\leq h^{k_g} \|v \otimes \nabla_{\Gamma_h}\|_{\Gamma_h} \|D_\Gamma w^l\|_{\Gamma_h} + h^{k_g} \|D_{\Gamma_h} v\|_{\Gamma_h} \|w \otimes \nabla_{\Gamma_h}\|_{\Gamma_h} \quad (5.110)$$

$$\leq h^{k_g} \|v \otimes \nabla_{\Gamma_h}\|_{\Gamma_h} \|w \otimes \nabla_{\Gamma_h}\|_{\Gamma_h} \quad (5.111)$$

Combining the estimates we arrive at

$$Q_a(v, w) \lesssim h^{k_g} \|v \otimes \nabla_{\Gamma_h}\|_{\Gamma_h} \|w \otimes \nabla_{\Gamma_h}\|_{\Gamma_h} \quad (5.112)$$

**Estimate (5.102).** Changing domain of integration from  $\Gamma$  to  $\Gamma_h$  we obtain

$$Q_l(v) = (f, v)_\Gamma - (f, v)_{\Gamma_h} \quad (5.113)$$

$$= ((|B| - 1)f, v)_{\Gamma_h} \quad (5.114)$$

$$\lesssim h^{k_g+1} \|f\|_{\Gamma_h} \|v\|_{\Gamma_h} \quad (5.115)$$

$$\lesssim h^{k_g+1} \|f\|_\Gamma \|v\|_{\Gamma_h} \quad (5.116)$$

where we used (4.14) followed by the norm equivalence (4.29).  $\square$

## 5.6 Error Estimates

**Theorem 5.1 (Energy Error Estimate)** *Let  $u$  be the solution to (2.18) and  $u_h$  the solution to (3.19), and assume that the geometry approximation assumptions are fulfilled and  $k_g \geq 1$ , then the following estimate holds*

$$\|e\|_h \lesssim h^{k_u} \|f\|_{H_{\tan}^{k_u-1}(\Gamma)} + h^{k_g-1} \|f\|_\Gamma \quad (5.117)$$

for all  $h \in (0, h_0]$  and  $h_0$  small enough.

**Proof.** Let  $e = u - \pi_h u + \underbrace{\pi_h u - u_h}_{e_h}$  and note that

$$\|e\|_h \leq \|u - \pi_h u\|_h + \|e_h\|_h \quad (5.118)$$

$$\lesssim h^{k_u} \|u\|_{H_{\tan}^{k_u+1}(\Gamma)} + \|e_h\|_h \quad (5.119)$$

To estimate  $\|e_h\|_h$  we add and subtract suitable terms

$$\|e_h\|_h^2 = A_h(e_h, e_h) \quad (5.120)$$

$$= A_h(\pi_h u - u, e_h) + A_h(u - u_h, e_h) \quad (5.121)$$

$$= A_h(\pi_h u - u, e_h) + A_h(u, e_h) - l_h(e_h) \quad (5.122)$$

$$= A_h(\pi_h u - u, e_h) + a_h(u, e_h) - a(u, e_h) + l(e_h) - l_h(e_h) + s_h(u, e_h) \quad (5.123)$$

$$= A_h(\pi_h u - u, e_h) - Q_a(u, e_h) + Q_l(e_h) + s_h(u, e_h) \quad (5.124)$$

$$\begin{aligned} &\lesssim \|\pi_h u - u\|_h \|e_h\|_h + h^{k_g} \|u\|_{H^1(\Gamma)} \|e_h\|_h \\ &\quad + h^{k_g+1} \|f\|_\Gamma \|e_h\|_h + h^{k_g-1} \|u\|_\Gamma \|e_h\|_h \end{aligned} \quad (5.125)$$

where we used Lemma 5.8, equivalence of norms (4.29), and the estimate below which follows from the Cauchy–Schwarz inequality and geometric error bounds

$$s_h(u, e_h) \leq \|u\|_{s_h} \|e_h\|_{s_h} \quad (5.126)$$

$$= h^{-1} \|n_h \cdot u\|_{\Gamma_h} \|e_h\|_{s_h} \quad (5.127)$$

$$= h^{-1} \|(n_h - n) \cdot u\|_{\Gamma_h} \|e_h\|_{s_h} \quad (5.128)$$

$$\lesssim h^{-1} \|n - n_h\|_{L^\infty(\Gamma_h)} \|u\|_{\Gamma_h} \|e_h\|_{s_h} \quad (5.129)$$

$$\lesssim h^{k_g-1} \|u\|_\Gamma \|e_h\|_{s_h} \quad (5.130)$$

Finally, using the interpolation error estimate (5.89) and the elliptic shift estimate (2.20) we obtain

$$\|e_h\|_h \lesssim h^{k_u} \|u\|_{H_{\tan}^{k_u+1}(\Gamma)} + h^{k_g} \|u\|_{H_{\tan}^{k_g}(\Gamma)} + h^{k_g-1} \|u\|_\Gamma + h^{k_g+1} \|f\|_\Gamma \quad (5.131)$$

$$\lesssim h^{k_u} \|f\|_{H_{\tan}^{k_u-1}(\Gamma)} + h^{k_g-1} \|f\|_\Gamma \quad (5.132)$$

which concludes the proof.  $\square$

**Theorem 5.2 ( $L^2$  Error Estimate)** *Under the same assumptions as in Theorem 5.1 and  $k_g \geq 2$  the following estimate holds*

$$\|e\|_\Gamma \lesssim h^{k_u+1} \|f\|_{H_{\tan}^{k_u-1}(\Gamma)} + h^{k_g} \|f\|_\Gamma \quad (5.133)$$

**Proof.** Splitting the error in a tangential and normal part

$$\|e\|_\Gamma \leq \|e_t\|_\Gamma + \|e_n\|_\Gamma \quad (5.134)$$

Here we have the following estimate of the normal component

$$\|e_n\|_\Gamma \lesssim \|e \cdot n\|_{\Gamma_h} \quad (5.135)$$

$$\lesssim \|e \cdot (n - n_h)\|_{\Gamma_h} + \|e \cdot n_h\|_{\Gamma_h} \quad (5.136)$$

$$\lesssim h^{k_g} \|e\|_{\Gamma_h} + \|e \cdot n_h\|_{\Gamma_h} \quad (5.137)$$

$$\lesssim h^{k_g} \|e\|_\Gamma + h \|e\|_h \quad (5.138)$$

$$\lesssim h^{k_g} \|e_t\|_\Gamma + h^{k_g} \|e_n\|_\Gamma + h \|e\|_h \quad (5.139)$$

Using kickback and the energy norm estimate we obtain

$$\|e_n\|_\Gamma \lesssim h^{k_g} \|e_t\|_\Gamma + h^{k_u+1} \|f\|_{H^{k_u-1}(\Gamma)} + h^{k_g} \|f\|_\Gamma \quad (5.140)$$

Next to estimate the tangential part of the error we introduce the dual problem: find  $\phi \in H_{\tan}^1(\Gamma)$  such that

$$a(v, \phi) = (v, \psi) \quad \forall v \in H_{\tan}^1(\Gamma) \quad (5.141)$$

where  $\psi \in L_{\tan}^2(\Gamma)$ . Using the elliptic shift property (2.20) we have

$$\|\phi\|_{H_{\tan}^2(\Gamma)} \lesssim \|\psi\|_\Gamma \quad (5.142)$$

Setting  $v = \psi = e_t$ , and adding and subtracting suitable terms we obtain

$$\|e_t\|_\Gamma^2 = (e_t, \psi)_\Gamma \quad (5.143)$$

$$= a(e_t, \phi) \quad (5.144)$$

$$= a(e_t, \phi - \pi_h \phi) + a(e_t, \pi_h \phi) \quad (5.145)$$

$$= a(e_t, \phi - \pi_h \phi) + l(\pi_h \phi) - a(u_{h,t}, \pi_h \phi) \quad (5.146)$$

$$= a(e_t, \phi - \pi_h \phi) + l(\pi_h \phi) - l_h(\pi_h \phi) \quad (5.147)$$

$$\begin{aligned} &+ A_h(u_h, \pi_h \phi) - a(u_h, \pi_h \phi) + a(u_{h,n}, \pi_h \phi) \\ &= a(e_t, \phi - \pi_h \phi) + Q_l(\pi_h \phi) - Q_a(u_h, \pi_h \phi) \end{aligned} \quad (5.148)$$

$$+ a(u_{h,n}, \pi_h \phi) + s_h(u_h, \pi_h \phi)$$

Using the Cauchy–Schwarz inequality, interpolation estimates, Lemma 5.8, Lemma 5.6 and bounds which we list and verify below we obtain

$$\|e_t\|_\Gamma^2 \lesssim \|e_t\|_a \|\phi - \pi_h \phi\|_a + h^{k_g+1} \|f\|_\Gamma \|\pi_h \phi\|_h \quad (5.149)$$

$$+ h^{k_g} \|u_h\|_h \|\pi_h \phi\|_h + \|u_{h,n}\|_a \|\pi_h \phi\|_a + \|u_h\|_{s_h} \|\pi_h \phi\|_{s_h}$$

$$\lesssim (h^{k_u+1} \|f\|_\Gamma + h \|e\|_h) \|\phi\|_{H_{\tan}^2(\Gamma)} + h^{k_g+1} \|f\|_\Gamma \|\phi\|_{H_{\tan}^1(\Gamma)} \quad (5.150)$$

$$+ h^{k_g} \|f\|_\Gamma \|\phi\|_{H_{\tan}^1(\Gamma)} + (h^{k_g} \|e_t\|_\Gamma + h \|e\|_h) \|\phi\|_{H_{\tan}^2(\Gamma)} + (h + h^{k_g-1}) \|e\|_h \|\phi\|_{H_{\tan}^2(\Gamma)}$$

$$\lesssim \left( h^{k_g} \|e_t\|_\Gamma + \underbrace{(h + h^{k_g-1}) \|e\|_h + (h^{k_u+1} + h^{k_g+1} + h^{k_g}) \|f\|_\Gamma}_{\star} \right) \underbrace{\|\psi\|_\Gamma}_{=\|e_t\|_\Gamma} \quad (5.151)$$

where we hide the  $h^{k_g} \|e_t\|_\Gamma$  term on the right with a kickback argument. Using the energy norm error estimate (5.117) we obtain

$$\star \leq (h + h^{k_g-1}) \left( h^{k_u} \|f\|_{H_{\tan}^{k_u-1}(\Gamma)} + h^{k_g-1} \|f\|_\Gamma \right) + (h^{k_u+1} + h^{k_g+1} + h^{k_g}) \|f\|_\Gamma \quad (5.152)$$

$$\lesssim (1 + h^{k_g-2}) h^{k_u+1} \|f\|_{H_{\tan}^{k_u-1}(\Gamma)} + (1 + h + h^{k_g-2} + h^{k_u+1-k_g}) h^{k_g} \|f\|_\Gamma \quad (5.153)$$

which, together with the bounds we verify below, completes the proof.

In the calculation above we used the following bounds on the error

$$\|e_t\|_a \lesssim h^{k_u} \|f\|_\Gamma + \|e\|_h \quad (5.154)$$

on the discrete solution

$$\|u_h\|_h \lesssim \|f\|_\Gamma \quad (5.155)$$

$$\|u_{h,n}\|_a \lesssim \|e_n\|_\Gamma \lesssim h^{k_g} \|e_t\|_\Gamma + h \|e\|_h \quad (5.156)$$

$$\|u_h\|_{s_h} \lesssim \|e\|_h + h^{k_g-1} \|f\|_\Gamma \quad (5.157)$$

and on the interpolant

$$\|\pi_h \phi\|_h \lesssim \|\phi\|_{H^1(\Gamma)} \lesssim \|\phi\|_{H_{\tan}^2(\Gamma)} \quad (5.158)$$

$$\|\pi_h \phi\|_a \lesssim h \|\phi\|_{H^2(\Gamma)} + \|\phi\|_{H_{\tan}^1(\Gamma)} \lesssim \|\phi\|_{H_{\tan}^2(\Gamma)} \quad (5.159)$$

$$\|\pi_h \phi\|_{s_h} \lesssim (h + h^{k_g-1}) \|\phi\|_{H^2(\Gamma)} \lesssim (h + h^{k_g-1}) \|\phi\|_{H_{\tan}^2(\Gamma)} \quad (5.160)$$

**Verification of (5.154).** This bound is established by adding and subtracting suitable terms, using the identity  $Pe_n \otimes \nabla_\Gamma = (e \cdot n)\kappa$  and interpolation estimates

$$\|e_t\|_a \lesssim \|e\|_a + \|e \cdot n\|_{\mathcal{K}_h} \quad (5.161)$$

$$\leq \|e_h\|_a + \|u - \pi_h u\|_a + \|e \cdot n\|_{\mathcal{K}_h} \quad (5.162)$$

$$\lesssim \|e_h\|_a + (h^{k_u} + h^{k_u+1}) \|u\|_{H^2(\Gamma)} \quad (5.163)$$

$$\lesssim \|e_h\|_{H^1(\Gamma)} + (h^{k_u} + h^{k_u+1}) \|f\|_\Gamma \quad (5.164)$$

where  $e_h = \pi_h u - u_h$  as above. By equivalence of norms (4.29) and Lemma 5.6 we have

$$\|e_h\|_{H^1(\Gamma)} \lesssim \|e_h\|_{H^1(\Gamma_h)} \lesssim \|e_h\|_h \quad (5.165)$$

and note in the proof of the energy estimate above that the bound for  $\|e_h\|_h$  is equivalent to the bound for  $\|e\|_h$ .

**Verification of (5.155).** This bound directly follows from the Cauchy–Schwarz inequality and Lemma 5.5.

**Verification of (5.156).** By the identity  $Pu_{h,n} \otimes \nabla_\Gamma = (u_h \cdot n)\kappa$  the bound follows

$$\|u_{h,n}\|_a = \|Pu_{h,n} \otimes \nabla_\Gamma\|_\Gamma = \|(u_h \cdot n)\kappa\|_\Gamma \lesssim \|e_n\|_\Gamma \quad (5.166)$$

and we then use the bound for  $\|e_n\|_\Gamma$  derived in the proof of the energy estimate.

**Verification of (5.157).** By the triangle inequality and interpolation we have

$$\|u_h\|_{s_h} \leq \|u - u_h\|_{s_h} + \|u\|_{s_h} \lesssim \|e\|_h + h^{k_g-1} \|u\|_\Gamma \lesssim \|e\|_h + h^{k_g-1} \|f\|_{H_{\tan}^{-1}(\Gamma)} \quad (5.167)$$

**Verification of (5.158).** By the triangle inequality and interpolation we have

$$\|\pi_h \phi\|_h^2 = \|P_h(\pi_h \phi) \otimes \nabla_{\Gamma_h}\|_{\Gamma_h}^2 + h^{-2} \|n_h \cdot \pi_h \phi\|_{\Gamma_h}^2 \quad (5.168)$$

$$\leq \|\phi \otimes \nabla_\Gamma\|_\Gamma^2 + h^{-2} \|n_h \cdot (\pi_h \phi - \phi)\|_{\Gamma_h}^2 + h^{-2} \|(n_h - n) \cdot \phi\|_{\Gamma_h}^2 \quad (5.169)$$

$$\leq (1 + h^{2k_g-2}) \|\phi\|_{H^2(\Gamma)}^2 \quad (5.170)$$



**Verification of (5.159).** Adding and subtracting terms and an interpolation estimate yields

$$\|\pi_h \phi\|_a \leq \|\phi - \pi_h \phi\|_a + \|\phi\|_a \lesssim h \|\phi\|_{H^2(\Gamma)} + \|\phi\|_{H_{\tan}^1(\Gamma)} \lesssim \|\phi\|_{H_{\tan}^2(\Gamma)} \quad (5.171)$$

**Verification of (5.160).** The last bound is established as follows

$$\|\pi_h \phi\|_{s_h} = h^{-1} \|n_h \cdot \pi_h \phi\|_{s_h} \leq h^{-1} \|(n_h - n) \cdot \pi_h \phi\|_{\Gamma} + h^{-1} \|n \cdot (\pi_h \phi - \phi)\|_{\Gamma} \quad (5.172)$$

$$\leq h^{k_g-1} \|\pi_h \phi\|_{\Gamma} + h \|\phi\|_{H^2(\Gamma)} \leq (h + h^{k_g-1}) \|\phi\|_{H^2(\Gamma)} \quad (5.173)$$

where we added and subtracted suitable terms, used the fact that  $\phi$  is tangential, an interpolation estimate, the  $L^2$  stability of  $\pi_h$ .  $\square$

**Remark 5.1** As noted in Remark 3.1 we, depending on the available geometry description, could use a better normal approximation in the form  $s_h$  which weakly enforces the tangent condition. For example, using the Lagrange interpolant  $\tilde{n}_h$  of the the exact normal, we have the estimate

$$\|n - \tilde{n}_h\|_{L^\infty(\Gamma_h)} \lesssim h^{k_g+1} \quad (5.174)$$

which is one order better compared to (3.1). This choice leads to the improved energy norm error estimate

$$\|e\|_h \lesssim h^{k_u} \|f\|_{H_{\tan}^{k_u-1}(\Gamma)} + h^{k_g} \|f\|_{\Gamma} \quad (5.175)$$

since the bound (5.126)–(5.130) is improved. The  $L^2$  estimate would however not be improved due to the loss of order in the geometry error estimate, see Lemma 5.8.

**Remark 5.2** The essential difference in the analysis of the symmetric formulation (2.21) is that we need to use Korn's inequality on surfaces. In this case there may also be a finite dimensional kernel consisting of so called Killing vector fields which may be taken into account using a quotient space. For surfaces with rotational symmetries restrictions of three dimensional rigid body rotations may induce a Killing field on the surface. A general approach, however, is to compute the kernel of the stiffness matrix numerically by solving an eigenvalue problem. We will return to these issues in forthcoming work on membranes, shells, and flow on surfaces.

## 6 Numerical Results

### 6.1 Model Problem and Numerical Example

**Geometry.** The surface of a torus can be expressed in Cartesian coordinates  $\{x, y, z\}$  as

$$\{x = (R + r \cos(\theta)) \cos(\phi), y = (R + r \cos(\theta)) \sin(\phi), z = r \sin(\theta)\} \quad (6.1)$$

where  $0 \leq \theta, \phi < 2\pi$  are angles and  $R, r > 0$  are fixed radii. For our model problem we consider such a geometry with radii  $R = 1$  and  $r = 0.6$ . Any point on the torus surface can thus be specified using the toroidal coordinates  $\{\theta, \phi\}$ .

**Manufactured Problem.** We manufacture problems on this geometry from the following ansatz as our analytical tangential vector field solution (expressed in Cartesian coordinates)

$$u = \begin{bmatrix} -r \sin(3\phi + \theta) \cos(\phi)^2 \sin(\theta) - \cos(\phi + 3\theta) \sin(3\phi) \sin(\phi)(R + r \cos(\theta)) \\ \cos(\phi + 3\theta) \sin(3\phi) \cos(\phi)(R + r \cos(\theta)) - r \sin(3\phi + \theta) \cos(\phi) \sin(\phi) \sin(\theta) \\ r \sin(3\phi + \theta) \cos(\phi) \cos(\theta) \end{bmatrix} \quad (6.2)$$

and we calculate the corresponding load tangential vector field for both the standard formulation and the symmetric formulation of the vector Laplacian. The analytical solution is illustrated in Figure 1(a).

**Nullspace in Symmetric Formulation.** In the case of the symmetric formulation on the torus surface will have a nullspace depicted in Figure 1(b). This is easily realized from the fact that the symmetric covariant derivative tensor is the projection of the symmetric Euclidean gradient, i.e. the linearized strain tensor, onto the tangential plane. Thereby the nullspace will consist of all linearized rigid body motions which are tangential to the surface. By searching for solutions orthogonal to this nullspace the problem is well posed. In the case of the standard formulation we do not have any nullspace by Lemma 5.2.

**The Mesh.** In Figure 2(a) we illustrate an example mesh of the model problem geometry. Higher order geometry interpolations are constructed by adding nodes for higher order Lagrange basis functions on each facet and mapping these nodes onto the torus surface  $\Gamma$  by a closest point map. To investigate whether or not convergence is dependent of the mesh structure we also use perturbed meshes, for example as illustrated in Figure 2(b), which are generated by randomly moving the mesh vertices a distance proportional to  $h$  and then mapping the vertices back onto  $\Gamma$  by a closest point map.

**A Numerical Example.** A numerical solution to the model problem using the symmetric formulation of the vector Laplacian is shown in Figure 3(a). In Figures 3(b)–(d) we present the magnitude of the error over  $\Gamma_h$  using different orders of geometry interpolation and as shown in the analysis above we here see the benefit of choosing the order of the geometry approximation one order above the order of the finite element approximation. Choosing an even higher order of geometry approximation gives no visual improvement.

## 6.2 Convergence

We perform convergence studies in  $L^2(\Gamma_h)$  norm on the model problem for both the standard formulation and the symmetric formulation. For both formulations results are presented using an equal order approximation of the geometry and the finite element space ( $k_g = k_u$ ) as well as using a geometry approximation which is one order higher than the finite element approximation ( $k_g = k_u + 1$ ). The results for the respective formulations are presented in Figures 4 and 5. To detect mesh dependence we also give results for the

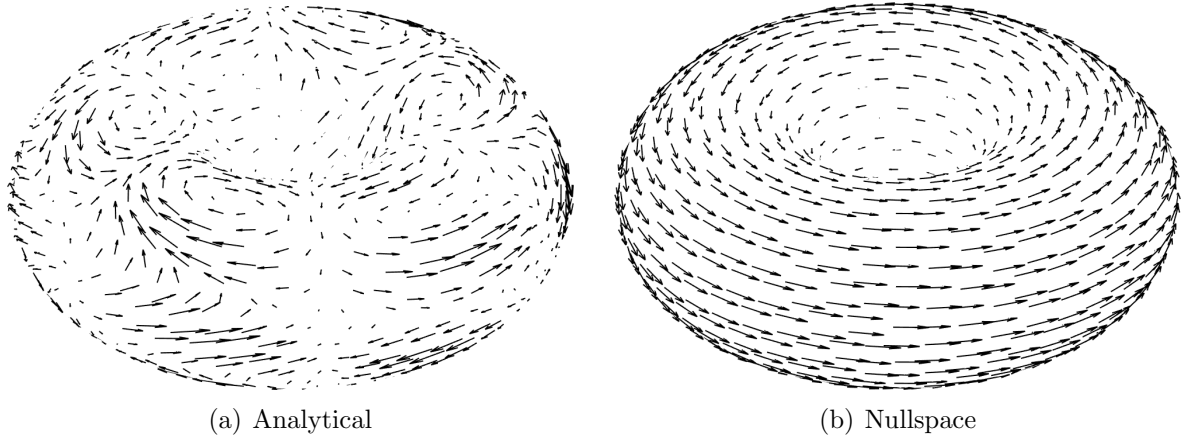


Figure 1: (a) Illustration of the analytical solution (6.2) to the model problem. (b) Illustration of the nullspace of the symmetric formulation of the vector Laplacian on the torus surface.

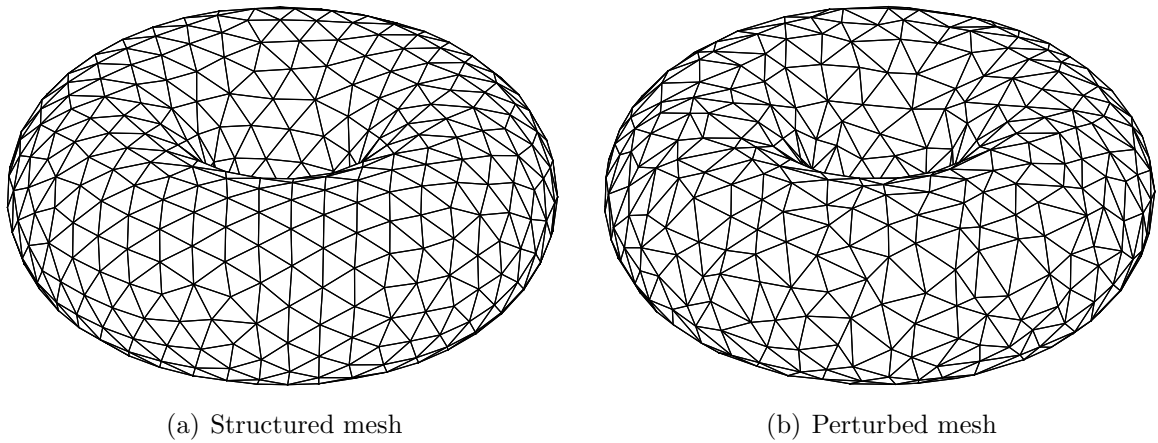


Figure 2: (a) Example of a structured mesh (a) with mesh size  $h = 0.25$  and a perturbed version of the same mesh (b).

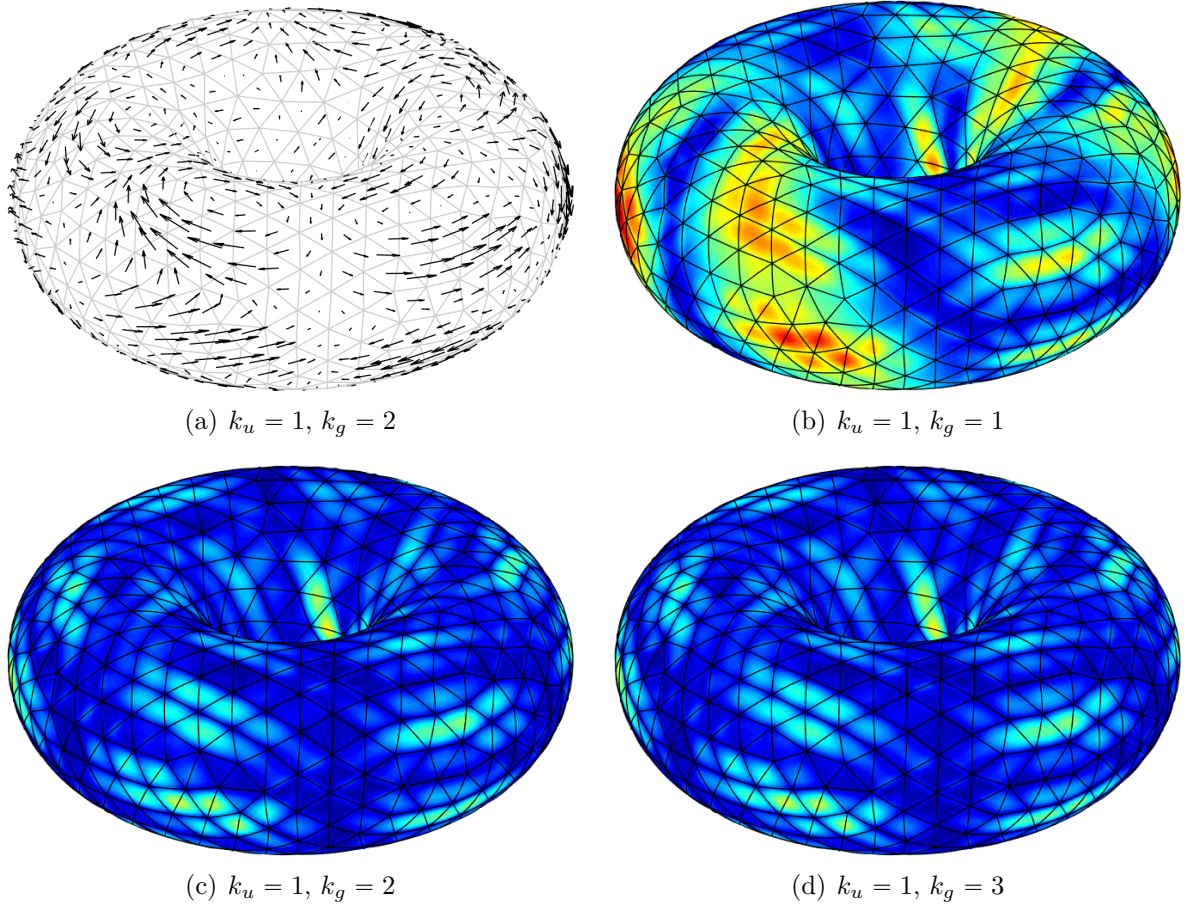
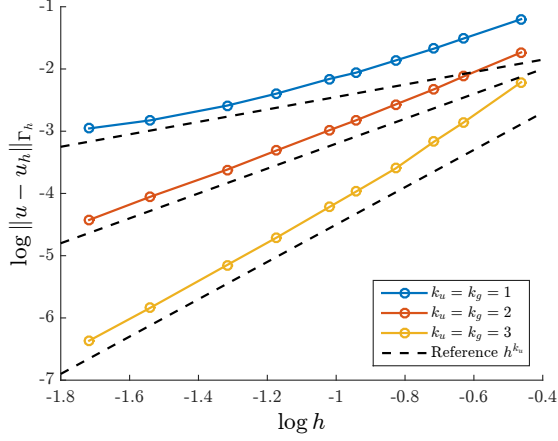


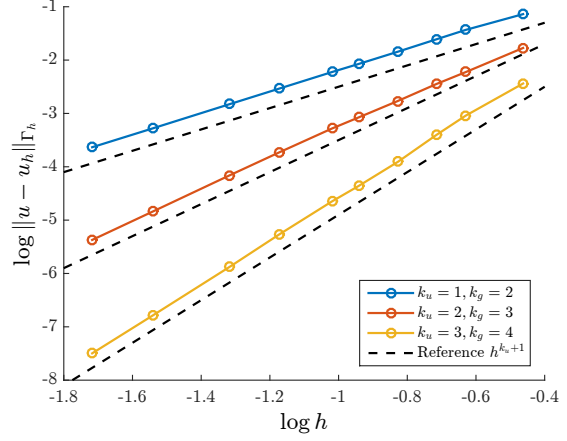
Figure 3: Numerical solution and error magnitudes in the model problem using linear finite elements ( $h = 0.25$ ,  $\beta = 100$ ). (a) Numerical solution from the symmetric formulation when the geometry is interpolated by piecewise quadratics. (b)–(d) Error magnitudes  $\|u - u_h\|_{\mathbb{R}^3}$  where blue is small and red is large using piecewise linear, quadratic respectively cubic geometry interpolation. Note that increasing the order of geometry approximation more than one step over the order of finite element approximation as in (d) yields visually identical results to (c).

standard formulation on perturbed meshes in Figure 6. All convergence results validate the conclusion from the analysis that the geometry approximation needs to be one order higher than the finite element approximation to achieve  $h^{k_u+1}$  convergence in  $L^2$  norm.

**Choice of  $\beta$ .** In the above result we have consistently used the normal penalty parameter  $\beta = 100$ . That this is a reasonable choice is motivated by the numerical study presented in Figure 7. However, in this study we note no particular sensitivity in the choice of this parameter.

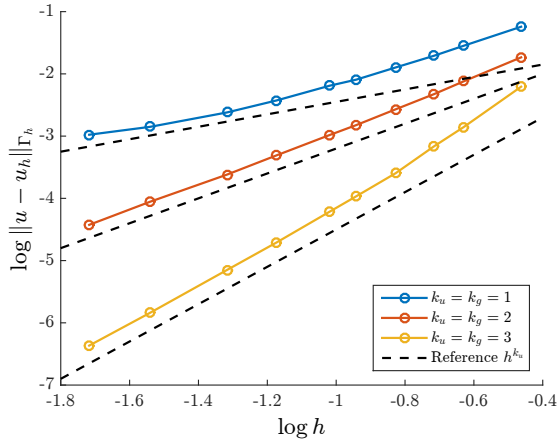


(a)  $k_g = k_u$

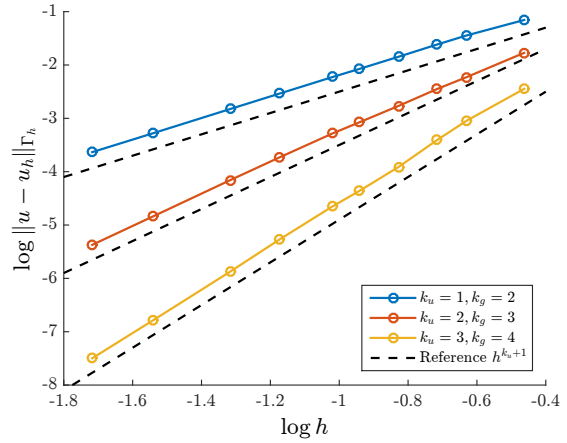


(b)  $k_g = k_u + 1$

Figure 4: Convergence studies in  $L^2(\Gamma_h)$  norm for the standard formulation of the vector Laplacian ( $\beta = 100$ ). In the left figure the same order of approximation is used for geometry and finite element approximation resulting in an error bounded by the geometry approximation. In the right figure one extra order is used for the geometry approximation.



(a)  $k_g = k_u$



(b)  $k_g = k_u + 1$

Figure 5: Convergence studies in  $L^2(\Gamma_h)$  norm for the symmetric formulation of the vector Laplacian ( $\beta = 100$ ). In the left figure the same order of approximation is used for geometry and finite element approximation resulting in an error bounded by the geometry approximation. In the right figure one extra order is used for the geometry approximation.

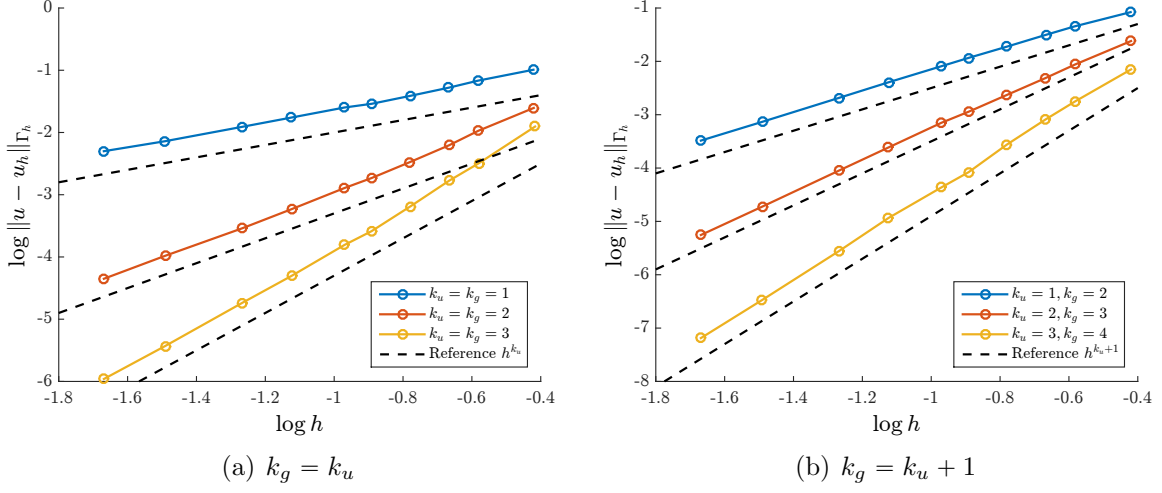


Figure 6: Convergence studies on perturbed meshes in  $L^2(\Gamma_h)$  norm for the standard formulation of the vector Laplacian ( $\beta = 100$ ). In the left figure the same order of approximation is used for geometry and finite element approximation resulting in an error bounded by the geometry approximation. In the right figure one extra order is used for the geometry approximation.

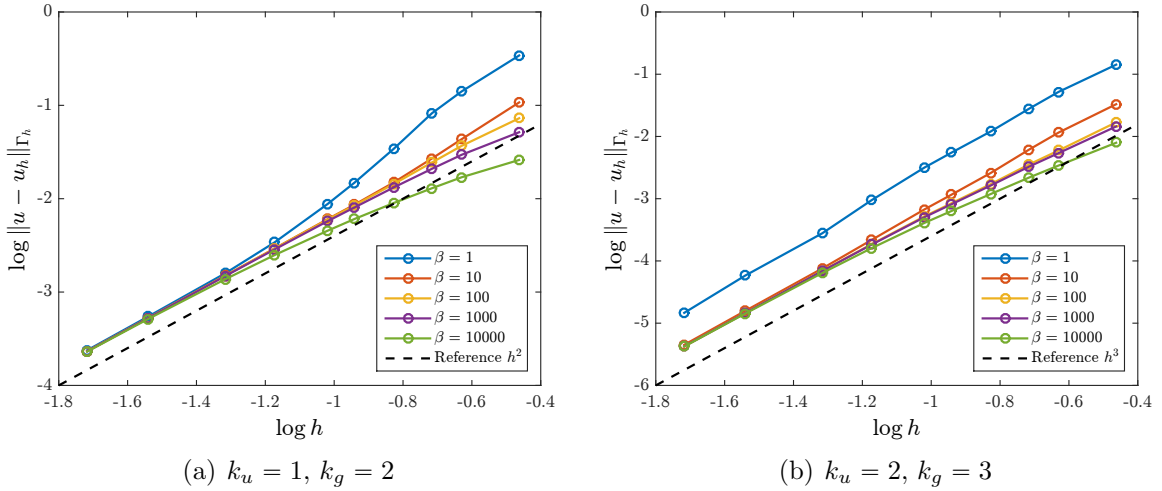


Figure 7: Convergence studies in  $L^2(\Gamma_h)$  norm on the model problem for the standard formulation using various values of the normal penalty parameter  $\beta$ .

## References

- [1] S. C. Brenner and L. R. Scott. *The Mathematical Theory of Finite Element Methods*, volume 15 of *Texts in Applied Mathematics*. Springer, New York, third edition, 2008.
- [2] E. Burman, P. Hansbo, and M. G. Larson. A stabilized cut finite element method for partial differential equations on surfaces: the Laplace-Beltrami operator. *Comput. Methods Appl. Mech. Engrg.*, 285:188–207, 2015.
- [3] A. Demlow. Higher-order finite element methods and pointwise error estimates for elliptic problems on surfaces. *SIAM J. Numer. Anal.*, 47(2):805–827, 2009.
- [4] M. P. do Carmo. *Riemannian geometry*. Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston, MA, 1992. Translated from the second Portuguese edition by Francis Flaherty.
- [5] G. Dziuk. Finite elements for the Beltrami operator on arbitrary surfaces. In *Partial differential equations and calculus of variations*, volume 1357 of *Lecture Notes in Math.*, pages 142–155. Springer, Berlin, 1988.
- [6] G. Dziuk and C. M. Elliott. Finite element methods for surface PDEs. *Acta Numer.*, 22:289–396, 2013.
- [7] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.
- [8] D. Gilbarg and N. S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.
- [9] P. Hansbo and M. G. Larson. Finite element modeling of a linear membrane shell problem using tangential differential calculus. *Comput. Methods Appl. Mech. Engrg.*, 270:1–14, 2014.
- [10] P. Hansbo and M. G. Larson. A stabilized finite element method for the darcy problem on surfaces. *IMA. J. Numer. Anal.*, 2016.
- [11] P. Hansbo, M. G. Larson, and F. Larsson. Tangential differential calculus and the finite element modeling of a large deformation elastic membrane problem. *Comput. Mech.*, 56(1):87–95, 2015.
- [12] M. Holst and A. Stern. Geometric variational crimes: Hilbert complexes, finite element exterior calculus, and problems on hypersurfaces. *Found. Comput. Math.*, 12(3):263–293, 2012.
- [13] K. Larsson and M. G. Larson. A continuous/discontinuous Galerkin method and a priori error estimates for the biharmonic problem on surfaces. *Math. Comput.*, in press.



- [14] J.-C. Nédélec. Curved finite element methods for the solution of singular integral equations on surfaces in  $R^3$ . *Comput. Methods Appl. Mech. Engrg.*, 8(1):61–80, 1976.
- [15] J. A. Thorpe. *Elementary topics in differential geometry*. Undergraduate Texts in Mathematics. Springer-Verlag, New York, 1994. Corrected reprint of the 1979 original.